The completeness of Real Numbers:

$$\mathbb{N} = \{0, 1, 2, 3, \cdots\}$$

"subtraction"

$$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$$

2

$\frac{1}{2} \notin \mathbb{Z}$

$\mathbb{Q} = +, \times$

$3a = 1$

$$\mathbb{Q} = \{\tfrac{p}{q} \text{ where } p, q \in \mathbb{Z} \text{ and } q \neq 0\}$$

$a+b = b+a$

$ab = ba$

$a \circ b = b \circ a$

$ab = 1$

$ba = 1$

$a(b+c) = ab + ac$

additive inverse

$\frac{p}{q} + \left(-\frac{p}{q}\right) = 0 \to$ Additive identity

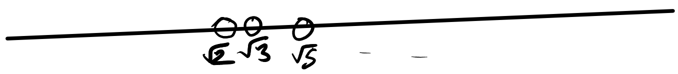
If $\frac{p}{q} \neq 0$, then $\frac{p}{q}$ has a multiplicative inverse

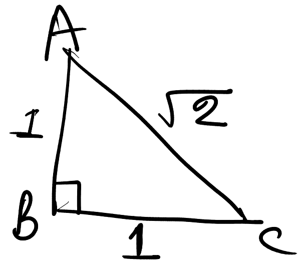
$\frac{p}{q} \times \frac{q}{p} = 1 \to$ multiplicative identity

$\mathbb{Q}$ is an ordered field

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$

$\sqrt{2}, \sqrt{3}, \sqrt{5}$





$\sqrt{2} \notin \mathbb{Q}$ $\qquad$ $\sqrt{2} = \frac{p}{q}$

$$\sqrt{2} = 1.41421356237\cdots$$

$S = \{1.4, 1.41, 1.414, 1.4142, \cdots\}$

$1.4 = \frac{14}{10}$

$1.41 = \frac{141}{100}$

$1.414 = \frac{1414}{1000}$

2

1

1.5

1.45

$\frac{1}{4} < \frac{1}{2} < \frac{2}{3}$ $\boxed{2+3i > 3+5i}$

The axiom of Completeness: Every nonempty set of real numbers that is bounded above has a least upper bound. $\rightarrow$ supremum of the set

$A \subseteq \mathbb{R}$

$A = \{a_1, a_2, \ldots\}$

Let b be an upper bound of A. That means

$a_i \leq b$

$\leq$

l is an upper bound of A and $l \leq b$ for any other upper bound b of A, then we call l the least upper bound of A.

$l = \sup A$

$B = \{b_1, b_2, \ldots\}$

lower bound $\rightarrow$ $c \leq b_i$ for all i

$c \leq g$

$g = \inf(B)$

$\hookrightarrow$ greatest lower bound

$S = \{\frac{1}{n} : n \in \mathbb{N}_{>0}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

≤ 1

$0 < \varepsilon$

$0 < \frac{1}{n} < \varepsilon$

supremum $\neq$ maximum

infimimum $\neq$ minimum

$\frac{1}{n+1} < \frac{1}{n}$

$(0, 2)$

$[1, 3]$

Lemma: Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then $s = \sup A$ if and only if for every choice of $\varepsilon > 0$, there exists an element $a \in A$ so that

$$\boxed{s - \varepsilon < a}$$

$a \leq s - \varepsilon$ for all $a \in A$

$s - \varepsilon < s$

s

NIP, BW, CC

Nested Interval Property: $I_n := [a_n, b_n]$

$[a, b]$

$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$

$I_n \supseteq I_{n+1}$

Then, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ ↳ empty set

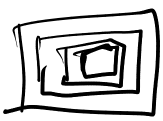
$$\boxed{A = \{a_n : n \in \mathbb{N}\}}$$

$I_1 = [a_1, b_1]$

$= \{x : a_1 \leq x \leq b_1\}$

$I_2 = [a_2, b_2]$

$= \{x : a_2 \leq x \leq b_2\}$



$a_1 < b_2$

$A = \{a_1, a_2, a_3, \cdots\}$

$[2, 3]$

$[2.5, 3]$

b_n

$a_1 < b_1$

$a_i \leq b_2$

$a_2 < b_2$

$I_2 \subseteq I_1 \Rightarrow$

$a_1 \leq a_2 \leq a_3 \leq \cdots$

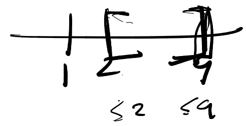
Since A is bounded above, by the AoC, sup A exists

let $\boxed{x} = \sup A$

$I_1 = [\cancel{2}, 4]$

$I_2 = [1, 2]$

$a_i \leq b_n \quad \forall i$



$a_1 \leq b_2$
$a_2 \leq b_2$
$a_3 \leq b_3 \leq b_2$

$a_1 < b_1$
$a_2 \leq b_2 \leq b_1$
$a_3 \leq b_3 \leq b_1$

$I_n = [a_n, b_n]$

$\boxed{a_n \leq x \leq b_n}$

$\Rightarrow x \in I_n$

$\boxed{x \leq b_n} \quad \forall n$

$x \in I_n \ \forall n \Rightarrow x \in \bigcap_{n=1}^{\infty} I_n$

Density of $\mathbb{Q}$ in $\mathbb{R}$: $\quad a < x < b$

(a, b)

$a < x_1 < x$
$a < x_2 < x_1$

Archimedean Property:

(i) Let $x \in \mathbb{R}$. Then there exists an $n \in \mathbb{N}$ so that
$n > x$.

(ii) Given any $\varepsilon > 0$. Then $\exists\, n \in \mathbb{N}$ so that
$0 < \frac{1}{n} < \varepsilon$

$a < x < b \qquad x = \frac{p}{q}$

$a \leq \frac{p}{q} \leq b \Rightarrow qa \leq p \leq qb$

Since $a<b$, $b-a>0 \Rightarrow \frac{1}{n}<b-a$

$\Rightarrow nb-na>1$

$\Rightarrow nb>na+1$

$\underline{m}>\underline{na}$,

$\subseteq \mathbb{N}$

$\underline{m}>na\geq m-1 \Rightarrow na+1\geq m$

$nb>na+1\geq \underline{m}>\underline{na}$

$b>\frac{m}{n}>a$

$a<\frac{m}{n}<b$

$a<x<b$ rational $\Rightarrow r=\frac{m}{n}$

$a-\sqrt{2}<r<b-\sqrt{2}$

$r+\sqrt{2}=x$ $\Rightarrow a<\underline{r+\sqrt{2}}<b$

$a<x<b$

If $r+\sqrt{2}\in\mathbb{Q} \Rightarrow r+\sqrt{2}=\frac{p}{q}$

$\Rightarrow \frac{m}{n}+\sqrt{2}=\frac{p}{q}$

$\Rightarrow \sqrt{2}=\frac{p}{q}-\frac{m}{n}$

$\Rightarrow \sqrt{2}\in\mathbb{Q}$

$1.4<x<\sqrt{2}$

$x<0<\sqrt{2}$

$0<0<\varepsilon$

Convergence of sequence:

A sequence (a_n) converges to a real number $\underline{a}$ if $\forall\, \varepsilon > 0$, $\exists$ an $N \in \mathbb{N}$ s.t. whenever $\boxed{n \geq N}$ it follows that $|a_n - a| < \varepsilon$

$\forall$ → for all

$\exists$ → there exists

$f : \mathbb{N}_{>0} \to \mathbb{R}$

$f(1) \to a_1$
$f(2) \to a_2$

$a_1, a_2, a_3, \ldots \ldots a$

0.1

< 0.1

For $\varepsilon = 0.1$, $N = 50$

a_{49} $(a_{50}$ a $)$
$a - 0.1$ $a + 0.1$

For $\varepsilon = 0.0001$
$N = 1000$

a

$(a + \varepsilon, a - \varepsilon)$

Example:

$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3}, \ldots$

$(a_n) = \left\{ \frac{1}{n} : n \in \mathbb{N}_{>0} \right\}$

$\frac{1}{m} < \frac{1}{n} < \varepsilon \quad (0, \varepsilon)$

$\left| \frac{1}{n} - 0 \right| < \varepsilon$

$n \geq N$

$\boxed{n = \boxed{N}}$

$\forall m > n$

$\frac{1}{n} < \varepsilon$

$\frac{1}{n} \in (0, \varepsilon)$

* Limit is unique here.
* Every convergent seq. is bounded but the converse is not true.

$\boxed{\mathbb{R}}$ → Hausdorff space

Monotone Convergence Theorem (MCT)

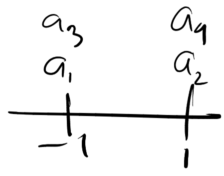
Monotone $\begin{cases} a_1 \le a_2 \le a_3 \le \cdots \\ a_1 \ge a_2 \ge a_3 \ge \cdots \end{cases}$

$-1, 1, -1, 1, -1, 1, \cdots$

$(-1, -1, -1, \ldots)$

$(1, 1, 1, \ldots, 1, \ldots)$

< 2



A bounded sequence is convergent if it is monotone.

NIP $\Rightarrow$ Bolzano–Weierstrass Theorem: Every bounded seq. contains a convergent subseq.
(BW)

<u>Subsequence:</u> (a_n) (a_{n_k})

$\{a_1, a_2, \cdots a_n, \cdots\}$

$\{a_1, a_3, a_5, a_{100}, \cdots\}$

a_1, a_1, a_2

<u>Thm:</u> Subsequences of a convergent sequence converge to the same limit.

$$(a_n) = \left\{\frac{1}{n^2} : n \in \mathbb{N}_{>0}\right\}$$

$$1 > \frac{1}{4} > \frac{1}{9} > \cdots > 0$$

Cauchy sequence:

(a_n) is Cauchy if $\forall \varepsilon > 0, \exists \underline{N} \in \mathbb{N}$ s.t. $\forall m, n \geq N$

$$|a_m - a_n| < \varepsilon$$

$$\varepsilon = 0.01$$

* Every convergent seq. is Cauchy

$$(x_n) \longrightarrow x$$

$n \geq 10$

let $\varepsilon > 0$. $\quad |x_n - x| < \varepsilon/2 \quad \forall n \geq N$ $\qquad m \geq 50$

$|x_m - x| < \varepsilon/2 \quad \forall m \geq M$

$|x_m - x_n| < \varepsilon$ $\qquad m, n > \max\{M, N\}$

$$|x_m - x_n| = |x_m - x + x - x_n| \leq |x_m - x| + |x_n - x|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$|a + b| \leq |a| + |b|$$

* Cauchy sequences are bounded.

Cauchy Criterion (CC): Convergent $\Leftrightarrow$ Cauchy

$\Rightarrow$

$\mathbb{Q}$ $\qquad$ $\sqrt{2}$

AoC, NIP, MCT, BW, CC

$$A_oC \Rightarrow NIP \Rightarrow BW \Rightarrow CC$$

$\hookrightarrow$ MCT

$\mathbb{Q}$ $\qquad \sqrt{2}$