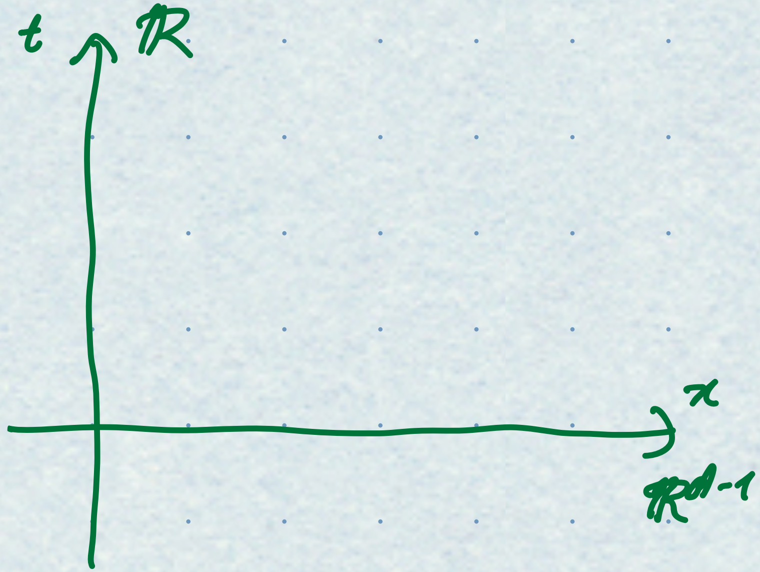
Operators on Hilbert Spaces

REF

- M. REED & B. SIMON, METHODS OF MODERN MATHEMATICAL PHYSICS - I
- K. SCHMÜDGEN, UNBOUNDED SA OP ON HILBERT SPACE
- D. BORTHWICK, SPECTRAL THEORY

MOTIVATION



WAVE EQ $\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u = 0$, $u \in C^\infty(\mathbb{R}^d, \overset{\mathbb{R}}{\mathbb{C}})$, $u(0,x) = f$

$\frac{\partial u}{\partial t}(0,x) = g$

loc

$$\Rightarrow u(t,x) = \cos(t\sqrt{-\Delta})f(x) + \sqrt{-\Delta}^{-1} \sin(t\sqrt{\Delta})g(x)$$

HEAT EQ $\left(\frac{\partial}{\partial t} - \Delta\right) u = 0$, $u(0,x) = f$

$$\Rightarrow u(t,x) = e^{t\Delta} f(x)$$

$C^k(\mathbb{R}^d, \mathbb{K})$, $\mathbb{K} := \mathbb{R}, \mathbb{C}$

SCHRÖDINGER EQ $\left(i\frac{\partial}{\partial t} - H\right) u = 0$, $u(0,x) = f$

$$\Rightarrow u(t,x) = e^{-itH} f(x)$$

$e^x = 1 + x + \frac{x^2}{2} + \cdots$ new def?

$f : A \longrightarrow \overset{A}{B} \longrightarrow$ $g : B \longrightarrow C$

$f^2 \not:= f \circ f$

$$A \xrightarrow{f} B \xrightarrow{g} C$$

x^2

$\frac{dx^2}{dx} = 2x$

$\Delta = \frac{\partial^2}{\partial x^2} : C^2(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$

$\cup \cancel{X}$

Dom Δ

$\frac{d^2 x^2}{dx^2} = 2$

WHAT IS $f(T)$?

CANONICAL COMMUTATION RELATIONS (CCRs) $[\hat{x}, \hat{p}] = i\hbar \mathbb{1}$

$A \qquad A_t$

$f : A \ni \xrightarrow{\text{SET}} \in B, \quad x \longmapsto y = f(x)$

$:=$

$\Rightarrow$

$A \longmapsto f(A) := ?$

$PDO^m(\mathbb{R}^d) := \{ \text{SPACE OF PDO OF ORDER } m \text{ ON } \mathbb{R}^d \}$

$\Delta \in PDO^2(\mathbb{R}^d), \qquad \Delta : C^2(\mathbb{R}^d) \longrightarrow$

$\Box \in PDO^2(\mathbb{R}^d) \qquad \Box : C^2(\mathbb{R}^d) \longrightarrow$

$H \in PDO^1(\mathbb{R}^d) \qquad H : C^1(\mathbb{R}^d) \longrightarrow$

$\mathbb{C}^d$

z, $\bar{\cdot} : z \mapsto \bar{z}$

$|z| := \sqrt{\bar{z} \cdot z}$

$(V, \mathbb{K} = \mathbb{C})$

NORMED VECTOR SPACE

$(V, \mathbb{C})$ ... COMPLEX VECTOR SPACE

$\| \cdot \| : V \longrightarrow \mathbb{R}$

$\forall \; u, v \in V, \quad c \in \mathbb{C}$:

(i) $\|v\| \geq 0$, $\|v\| = 0 \Leftrightarrow v = 0$

(ii) $\|cv\| = |c| \, \|v\|$, $\|u + v\| \leq \|u\| + \|v\|$

· <u>EXM</u> $K \subset \mathbb{R}^d$ CPT

$$C(K,\mathbb{R}) := \{ f : K \longrightarrow \mathbb{R} \quad \text{CONTINUOUS} \}$$

$$\|\cdot\| : C(K,\mathbb{R}) \to \mathbb{R}, \qquad f \mapsto \|f\| := \sup_{x \in K} |f(x)|$$

$$(C(K,\mathbb{R}), \|\cdot\|)$$

$$(V, \|\cdot\|)$$

$$\text{dist}(u,v) := \|u-v\|$$

$(u_n)_n \in V$ CONVERGES TO u IF

$$\lim_{n\to\infty} \|u_n - u\| = 0$$

$(v_n)_n \in V$ ... CAUCHY IF

$$\lim_{n,m\to\infty} \|v_n - v_m\| = 0$$

? DOES EVERY CAUCHY SEQUENCE IN V CONVERGE IN V ?

$\longrightarrow$ NO

IF YES $\Longrightarrow$ $(V, \|\cdot\|)$ ... COMPLETE AS A METRIC SPACE

· DEF BANACH SPACE := A COMPLETE NORMED VS

INNER PRODUCT SPACE

· DEF $(V, \mathbb{K})$

AN INNER PRODUCT ON V

$$\langle \cdot | \cdot \rangle : V \times V \longrightarrow \mathbb{K}, \quad (u,v) \longmapsto \langle u | v \rangle$$

$$\langle cu + v, w \rangle = \bar{c} \langle u | w \rangle + \langle v | w \rangle$$

$$\langle u | v \rangle = \overline{\langle v | u \rangle}$$

$$\langle v|v\rangle \geq 0$$

$$\langle v|v\rangle = 0 \iff v = 0$$

$(V, \langle\cdot|\cdot\rangle)$ ... INNER PRODUCT SPACE

·PROPERTIES

· CAUCHY-SCHWARTZ INEQUALITY

$$|\langle u|v\rangle| \leq \sqrt{\langle u|u\rangle}\,\sqrt{\langle v|v\rangle}$$

· $\|v\| := \sqrt{\langle v|v\rangle}$

· $$\langle u|v\rangle \overset{\mathbb{K}=\mathbb{C}}{=} \frac{1}{4}\left(\underbrace{\|u+v\|^2 - \|u-v\|^2}_{\mathbb{K}=\mathbb{R}} + i\,\|u+iv\|^2 - i\,\|u-iv\|^2\right)$$

· **DEF** HILBERT SPACE := COMPLETE INNER PRODUCT SPACE
$(\mathcal{H}, \langle \cdot | \cdot \rangle)$

$(V, \|\cdot\|, \widetilde{\|\cdot\|})$

$\|\cdot\|$ & $\widetilde{\|\cdot\|}$ ... EQUIVALENT IF

$\exists\ cst, \widetilde{cst} > 0 \mid cst\,\|\cdot\| \leq \widetilde{\|\cdot\|} \leq \widetilde{cst}\,\|\cdot\|$

$dist, \widetilde{dist}$

· **THM** $(V, \|\cdot\|, \widetilde{\|\cdot\|}) \rightsquigarrow (V, dist, \widetilde{dist})$

$\implies \|\cdot\|$ & $\widetilde{\|\cdot\|}$ ... EQUIVALENT $\iff$ $dist, \widetilde{dist}$ ... EQUIVALENT

$(\mathbb{K}, |\cdot|)$ ... COMPLETE $\rightsquigarrow$ BANACH SPACE

$\mathbb{K}^d$

$$\|z\|_1 := \sum_{i=1}^{d} |z_i|$$

$$\|z\|_p := \sqrt[p]{\sum_{i=1}^{d} |z_i|^p} \quad , \quad p = 1, 2, \ldots$$

$$\|z\|_\infty := \max_{i=1,\ldots,d} |z_i|$$

- THM $\dim V < \infty \implies \forall$ N.Q ON V ... EQUIVALENT
- COR ALL FINITE-DIMENSIONAL VS ... COMPLETE

SOME FUNCTIONAL SPACES

$\mathbb{K}^{\mathbb{N}} := \{ (\xi_k)_{k \in \mathbb{N}} \mid \forall k \in \mathbb{N} : \xi_k \in \mathbb{K} \}$

$\ell^{\infty}(\mathbb{K}) := \{ \xi \in \mathbb{K}^{\mathbb{N}} \mid (\xi_k)_{k \in \mathbb{N}} \cdots \text{ BOUNDED} \}$

$c(\mathbb{K}) := \{ \text{—"—} \mid \text{—"— —} \text{ CONVERGENT} \}$

$c_0(\mathbb{K}) := \{ \text{—"—} \mid \text{—"— —} \text{ NULL SEQUENCE} \}$

$c_e(\mathbb{K}) := \{ \text{—"—} \mid \text{—"— —} \text{ FINITE —"—} \}$

$U \subseteq \mathbb{R}^d$ SUBSET. $\operatorname{supp} f = \overline{\{ x \in U \mid f(x) \neq 0 \}}$

$\mathcal{B}(U) := \{ f : U \to \mathbb{K} \mid f \cdots \text{ BOUNDED} \}$

$$\|f\|_\infty := \sup_{x \in U} |f(x)|$$

$1 \leq p < \infty$

$$L^p(U, \mathbb{K}) := \left\{ [f] \in \dots \;\middle|\; \|f\|_p := \sqrt[p]{\int_U \underbrace{|f(x)|}_{x}^{\,p=2} dx} < \infty \right\}$$

$$L^\infty(\text{— " —}) := \left\{ \text{- - — " —} \;\; \|f\|_\infty := \operatorname*{ess\,sup}_{x \in U} |f(x)| < \infty \right\}$$

$$\operatorname*{ess\,sup} |f(x)| := \inf \{ M > 0 \mid x \in U : |f(x)| > M \text{ HAS LEBESGUE MEASURE } 0 \}$$

- EXM. $L^2(U, \mathbb{K})$ ... HILBERT SPACE $\quad L^2$ ... QM
- $\ell^p(\mathbb{K})$, $L^p(U, \mathbb{K})$, $p \neq 2$ ... NOT INNER PRODUCT SPACE
- $C(K, \mathbb{R})$, $\mathbb{K} \neq 0$ $\quad$ — " — $\quad$ ———
- $c_{cpt}(\mathbb{K})$ ... INNER PRODUCT SPACE BUT NOT HILBERT SPACE

· <u>DEF</u> $(\mathcal{H}, \langle\cdot|\cdot\rangle)$

ORTHOGONAL COMPLEMENT OF $V \subset \mathcal{H}$

$$V^{\perp} := \{u \in \mathcal{H} \mid \forall v \in V : \langle u|v\rangle = 0\}$$

$V \subset \mathcal{H}$ CLOSED $\Longrightarrow$ $\mathcal{H} = V \oplus V^{\perp}$

RECALL,

$(\mathcal{H}_1, \langle\cdot|\cdot\rangle_1)$, $(\mathcal{H}_2, \langle\cdot|\cdot\rangle_2)$ ... THEIR DIRECT SUM IS DEFINED AS THE SET

$$(u_1, u_2) \in \mathcal{H}_1 \times \mathcal{H}_2$$

$$\langle (u_1, u_2), (v_1, v_2)\rangle_{\mathcal{H}_1 \times \mathcal{H}_2} := \langle u_1 | v_1\rangle_1 + \langle u_2 | v_2\rangle_2$$

$$\bigoplus_{i=1}^{\infty} \mathcal{H}_i := \left\{(u_1, u_2, \dots) \mid u_i \in \mathcal{H}_i, \sum \|u_i\|_i^2 < \infty\right\}$$

• <u>DEF</u> $\mathcal{H}$ ... SEPARABLE IF IT ADMITS A COUNTABLE DENSE SEBSET

$U \subset \mathcal{H}$ DENSE $\qquad \bar{U} \qquad\qquad \bar{\mathbb{Q}} \qquad \mathbb{R}$

$\qquad \|\cdot\| \qquad\qquad\qquad \mathcal{H} \qquad\qquad\qquad\qquad |\cdot|$

$C^\infty(\mathbb{R}^d)$

• <u>DEF</u> $(e_1, e_2, \dots)$ SEQUENCE $\subset \mathcal{H}$

ORTHONORMAL IF $\langle e_i | e_j \rangle = \delta_{ij}$

• <u>THM</u> A SEPARABLE $\mathcal{H}$ ADMITS AN ORTHONORMAL BASIS.

OPERATORS

$(\mathcal{H}, \|\cdot\|_{\mathcal{H}}) \leftsquigarrow (\mathcal{H}, \langle\cdot|\cdot\rangle_{\mathcal{H}})$, $(\mathcal{K}, \|\cdot\|_{\mathcal{K}}) \leftsquigarrow (\mathcal{K}, \langle\cdot|\cdot\rangle_{\mathcal{K}})$

dom T (T, dom, $\mathcal{H}$)

LINEAR

· **DEF** $T : \mathcal{H} \to \mathcal{K}$, $u \mapsto T(u)$

$\mathcal{L}(\mathcal{H}, \mathcal{K})$

$$\|\cdot\| : \mathcal{L}(\mathcal{H}, \mathcal{K}) \to \mathbb{R}, \quad T \mapsto \|T\| := \sup_{u \in \mathcal{H} \setminus \{0\}} \frac{\|Tu\|_{\mathcal{K}}}{\|u\|_{\mathcal{H}}}$$

T ... BOUNDED IF $\|T\| < \infty$

$\mathcal{B}(\mathcal{H}, \mathcal{K})$

T ... UNBOUNDED IF IT IS NOT BOUNDED

EXM $f \in L^\infty(\mathbb{R}^d, \mathbb{K})$

MULTIPLICATION OP BY f

$$M_f : L^p(\mathbb{R}^d, \mathbb{K}) \longrightarrow L^p(\mathbb{R}^d, \mathbb{K}), \qquad u \longmapsto M_f(u) := f \cdot u$$

$$\lambda \cdot u \longmapsto M_f(\lambda u) = f \cdot (\lambda u) = \lambda \cdot f \cdot u \checkmark$$

$$\|M_f(u)\|_p = \|f \cdot u\|_p$$

$$\leq \|f\|_\infty \|u\|_p \qquad |f| \leq \|f\|_\infty \ldots$$

$$\Rightarrow \|M_f\| \leq \|f\|_\infty \checkmark$$

FOR $a < \|f\|_\infty$, SET $A := \{|f| \geq a\}$

$\mathbb{1}_A$ ... CHARACTERISTIC FUNCTION

$$\mathbb{1}_A : A \to \{0,1\}$$

$$x \mapsto \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

$$\|\mathbb{1}_A\|_p = \int_A 1 \, dx > 0$$

$$\|f \mathbb{1}_A\|_p \geq a \|\mathbb{1}_A\|_p$$

$$\hookrightarrow \forall a < \|f\|_\infty, \quad \|M_f\| \geq a$$

$$\boxed{\therefore \|M_f\| = \|f\|_\infty}$$

• <u>DEF</u> $T \in \mathcal{B}(\mathcal{H}, \tilde{\mathcal{H}})$, $S \in \mathcal{B}(\tilde{\mathcal{H}}, \mathcal{K})$

$$\circ : \mathcal{B}(\mathcal{H}, \tilde{\mathcal{H}}) \times \mathcal{B}(\tilde{\mathcal{H}}, \mathcal{K}) \longrightarrow \mathcal{B}(\mathcal{H}, \mathcal{K}), \quad (T, S) \longmapsto S \circ T$$

$$\|S \circ T\| \leq \|S\| \, \|T\|$$

• REM $(\mathcal{B}(\mathcal{H}), \circ)$ ... ALGEBRA
$\lambda T, \quad S+T, \quad S \circ T \in \mathcal{B}(\mathcal{H})$ $\Big\} \Longrightarrow$ Alg QM (AQM)
AQFT [HAAG]

$V \subset \mathcal{H}$ DENSE
$\overline{V} = \mathcal{H}$
$\|h - u\| < \varepsilon$

$= |\xi|^2$

• EXM $f : \mathbb{R}^d \longrightarrow \mathbb{K}$ " MEASURABLE "

$\mathrm{dom}(M_f) := \{ u \in L^2(\underset{K}{\mathbb{R}^d}, \mathbb{K}) \mid f \cdot u \in L^2(\underset{K}{\mathbb{R}^d}, \mathbb{K}) \}$

$M_f : \mathrm{dom}(M_f) \longrightarrow L^2(\mathbb{R}^d, \mathbb{K}), \quad u \longmapsto M_f(u) := f \cdot u$

M_f ... BOUNDED $\Longleftrightarrow f \in L^\infty(\mathbb{R}^d, \mathbb{K})$

$\notin$

$f \doteq 1 \Longrightarrow \|f\|_\infty = \infty$

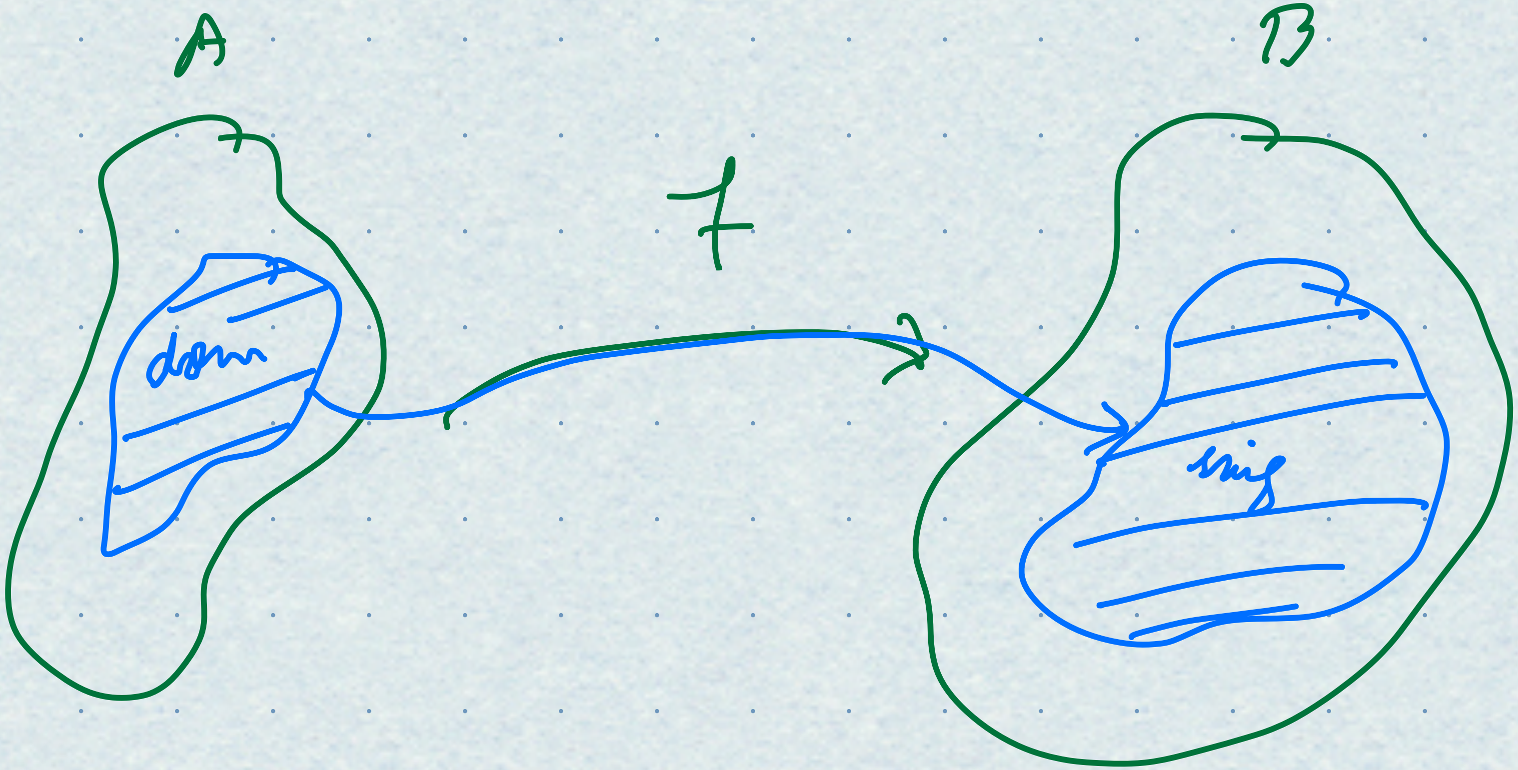
$K \subset \mathbb{R}^d$ CPT $\Longrightarrow \|f\|_\infty < \infty$

$= L^2(K, \mathbb{K})$

$\Longrightarrow M_f : \mathrm{dom}\, M_f \longrightarrow L^2(K, \mathbb{K})$ bounded

COMPACT SPACE $\longleftrightarrow$ SEQUENCE HAS CONVERGENT SUB-SEQ

COMPLETE $\longleftrightarrow$ $\exists$ SEQ THAT CONVERGES ... VECTOR



$(u_n)_n \longrightarrow u$

$T u_n \overset{?}{\longrightarrow} u$

· <u>DEF</u> $f \in L^1(\mathbb{R}^d, \mathbb{K})$

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i x \cdot \xi} \, f(x) \, dx$$

$$\mathcal{F} : L^1(\mathbb{R}^d, \mathbb{K}) \longrightarrow \, ?$$

SCHWARTZ SPACE $\mathcal{S}(\mathbb{R}^d) := \{ f \in C^\infty(\mathbb{R}^d, \mathbb{K}) \mid \forall \alpha, \beta \in \mathbb{N}_0^d : \|x^\alpha D^\beta f\|_\infty < \infty \}$

$\alpha = (\alpha_1, \dots, \alpha_d)$,

$|\alpha| := |\alpha_1| + \dots |\alpha_d|$, $\alpha := \alpha_1 \cdots \alpha_d$

MULTI-INDEX NOTATION

$$D^\beta := (-i)^{|\beta|} \frac{\partial^{|\beta|}}{\partial (x^1)^{\beta_1} \cdots \partial (x^d)^{\beta_d}}$$

$$u \in \mathcal{S}(\mathbb{R}^d, \mathbb{K}) \implies \widehat{(D_x^\alpha u)}(\xi) = (i\xi)^\alpha (\mathcal{F}u)(\xi)$$

$$\widehat{(x^\alpha u)}(\xi) = (iD_\xi)^\alpha (\mathcal{F}u)(\xi)$$

$$\implies \hat{\ } : \mathcal{S}(\mathbb{R}^d, \mathbb{K}) \longrightarrow \mathcal{S}(\mathbb{R}^d, \mathbb{K})$$

$$\overline{\mathcal{S}}(\mathbb{R}^d, \mathbb{K}) = L^2(\mathbb{R}^d, \mathbb{K})$$

· <u>DEF</u> $(\mathcal{H}, \|\cdot\|)$, $(\mathcal{K}, \|\cdot\|)$

$T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ ISOMETRY $\iff \|Tu\|_{\mathcal{K}} = \|u\|_{\mathcal{H}}$

$\|T\| = 1$

T ... BIJECTIVE IF $\exists\ T^{-1} \in \mathcal{B}(\mathcal{K}, \mathcal{H})$

$\notin$

$(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{H}})$, $(\mathcal{K}, \langle \cdot | \cdot \rangle_{\mathcal{K}})$

T ... UNITARY := BIJECTIVE & ISOMETRY

$\langle Tu | Tv \rangle = \langle u | v \rangle = \langle T^2 u | v \rangle$

$\|T\| = 1$

- **<u>THM [PLANCHEREL]</u>** $\mathcal{F} : L^2(\mathbb{R}^d, \mathbb{K}) \longrightarrow L^2(\mathbb{R}^d, \mathbb{K})$ UNITARY

$$\Delta = -\frac{\partial^2}{\partial x^2}$$

$|\psi\rangle$

$$\Longrightarrow \mathcal{F}(\Delta u) = |\xi|^2 \mathcal{F}(u) = M_{|\xi|^2} \mathcal{F}(u)$$

$$\mathrm{dom}(\Delta) := \left\{ u \in L^2(\mathbb{R}^d, \mathbb{K}) \mid |\xi|^2 \mathcal{F}(u) \in L^2(\mathbb{R}^d, \mathbb{K}) \right\}$$

$$\Delta : \mathrm{dom}(\Delta) \longrightarrow L^2(\mathbb{R}^d, \mathbb{K}), \quad u \longmapsto \Delta(u) := M_{|\xi|^2} \hat{u}$$

$\|M_{|\xi|^2}\| = \infty \Longrightarrow \Delta$ ... UNBOUNDED OP

$\hat{p} |p\rangle = p |p\rangle$

QM

Math rep universe