Approval

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Singularity Analysis in Schwarzschild Spacetime

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Abstract

This report reviews a mathematical analysis of singularities in Schwarzschild spacetime. We provide detailed computations of curvature invariants—Ricci scalar and Kretschmann scalar—demonstrating that the $r = 2\mu$ singularity is coordinate-dependent while $r = 0$ represents a genuine curvature singularity.After that, a calculation of null geodesic incompleteness is shown to prove that there exists a physical singularity at $r = 0$ for Schwarzschild manifold.

1 Preliminaries

Definition 1.1 (Lorentzian Manifold). *A Lorentzian manifold* (M, g) *is a smooth manifold* M *equipped with a metric tensor* g *of signature* $(-, +, +, +)$*; see e.g. [1, 2].*

Definition 1.2 (Schwarzschild Spacetime). *The Schwarzschild spacetime is given by [3] (see also, e.g. [1, 2])*

$$\begin{aligned} M &:= \mathbb{R} \times ((0, 2\mu) \cup (2\mu, \infty)) \times \mathbb{S}^2, && (1) \\ g &:= \left(-\left(1 - \frac{2\mu}{r}\right),\ \left(1 - \frac{2\mu}{r}\right)^{-1},\ r^2,\ r^2 \sin^2\theta \right) && (2) \end{aligned}$$

where $\mu \geq 0$ *is the mass parameter and we have used standard coordinates* (t, r, θ, ϕ) *with the convention* $G = c = 1$*.*

A curve on M is a smooth map $\gamma : I \to M$, where $I \subset \mathbb{R}$ is an interval. The tangent vector field along γ is the map $T : I \to TM$ given by $T(\lambda) = \gamma'(\lambda)$. In local coordinates $(x^1, \ldots, x^n)$, we write $\gamma(\lambda) = (x^1(\lambda), \ldots, x^n(\lambda))$ and

$$T = \frac{dx^\mu}{d\lambda}\partial_\mu.$$

Let ∇ be the Levi-Civita connection on M. For any vector field V defined along γ, the **covariant derivative** of V along γ is given by

$$\frac{DV}{d\lambda} = \nabla_T V = \left(\frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}V^\beta\right)\partial_\mu,$$

where $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols of the connection ∇.

Definition 1.3 (Christoffel Symbols)**.** *Let* (M, g) *be a Lorentzian manifold with metric tensor* g*. The Christoffel symbols of the second kind (also called the Levi-Civita connection coefficients) are defined in local coordinates* $\{x^\mu\}$ *by:*

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho} \right)$$

where $g_{\mu\nu}$ *are the components of the metric tensor,* $g^{\lambda\rho}$ *are the components of the inverse metric tensor, and we employ the Einstein summation convention over repeated indices.*

Definition 1.4 (Geodesic)**.** *A smooth curve* $\gamma : I \to M$ *is called a geodesic if its tangent vector field is parallel along* γ*:*

$$\nabla_T T = 0$$

Definition 1.5 (Affine Parameter)**.** *A parameter* λ *along a curve* γ *is called an affine parameter if there exists a reparameterization* $\lambda' = a\lambda + b$ *with* $a, b \in \mathbb{R}$*,* $a \neq 0$*, such that the tangent vector* $T = d\gamma/d\lambda$ *satisfies* $\nabla_T T = 0$*.*

In local coordinates,

$$\begin{aligned}
(\nabla_T T)^\mu &= \frac{dT^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} T^\alpha T^\beta \\
&= \frac{d}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} \\
&= \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}.
\end{aligned}$$

Since $\nabla_T T = 0$, each component must vanish identically. This yields the geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{3}$$

The analysis of singularities in general relativity requires sophisticated mathematical tools from differential geometry and global analysis [4, 5, 6, 1] (see also, e.g. [2, 7]. Unlike in classical field theory where singularities are typically defined as points where physical quantities diverge, in general relativity we adopt a more geometric approach based on the global structure of spacetime.

Definition 1.6 (Time-like Geodesic)**.** *A time-like geodesic is a smooth curve* $\gamma : I \to M$*, where* $I \subseteq \mathbb{R}$ *is an interval, satisfying the geodesic equation:*

$$\nabla_{\dot\gamma} \dot\gamma = 0$$

and with $g(\dot\gamma, \dot\gamma) < 0$ *along the curve.*

Definition 1.7 (Geodesic Completeness)**.** *A Lorentzian manifold* (M, g) *is said to be geodesically complete if every maximal geodesic* $\gamma : I \to M$ *is defined for all affine parameter values, i.e.,* $I = \mathbb{R}$*.*

Definition 1.8 (Geodesic Incompleteness)**.** *A Lorentzian manifold* (M, g) *is geodesically incomplete if there exists an inextendible geodesic* $\gamma : I \to M$ *that is not defined for all values of the affine parameter, i.e.,* $I \subsetneq \mathbb{R}$ *is a proper subset of the real line.*

Definition 1.9 (Inextendible Geodesic). *A geodesic* $\gamma : I \to M$ *is called inextendible if there is no geodesic* $\tilde{\gamma} : J \to M$ *with* $I \subsetneq J$ *such that* $\tilde{\gamma}|_I = \gamma$.

Definition 1.10 (Singular Spacetime). *A spacetime* (M, g) *is said to be singular if it contains at least one incomplete, inextendible geodesic. More precisely, the spacetime is singular if either [5, 6]:*

1. *There exists an incomplete, inextendible time-like geodesic, or*
2. *There exists an incomplete, inextendible null geodesic.*

Definition 1.11 (Causal Geodesic Incompleteness). *A spacetime is time (resp. null)-like geodesically incomplete if there exists a time (resp. null)-like geodesic* $\gamma : [0, a) \to M$ *(or* $\gamma : (a, 0] \to M$*) that is inextendible beyond the finite affine parameter value* a.

2 Singularity Analysis: Curvature Invariants

In the case of Schwarzschild spacetime (Definition 1.2), the metric g given by (2).

2.1 Christoffel Symbols for Schwarzschild Metric

We systematically compute all non-vanishing components:

Time component ($\sigma = t$)

For $\Gamma^t_{\mu\nu}$, the only non-zero component is symmetric in its lower indices:

$$\begin{aligned}\Gamma^t_{tr} = \Gamma^t_{rt} &= \frac{1}{2} g^{tt} \left(\partial_t g_{rt} + \partial_r g_{tt} - \partial_t g_{tr}\right) \\ &= \frac{1}{2}\left(-f(r)^{-1}\right)\left(0 + \partial_r(-f(r)) - 0\right) \\ &= \frac{1}{2f(r)} \partial_r f(r)\end{aligned}$$

Since $f(r) = 1 - \frac{2\mu}{r}$, we have $\partial_r f(r) = \frac{2\mu}{r^2}$, yielding:

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{\mu}{r^2 f(r)} \tag{4}$$

Radial component ($\sigma = r$)

For $\Gamma^r_{\mu\nu}$, we compute four distinct components:

$$
\begin{aligned}
\Gamma^r_{tt} &= \frac{1}{2} g^{rr} \left(\partial_t g_{tr} + \partial_t g_{rt} - \partial_r g_{tt}\right) \\
&= \frac{1}{2} f(r) \left(0 + 0 - \partial_r(-f(r))\right) \\
&= \frac{1}{2} f(r) \partial_r f(r) = \frac{\mu f(r)}{r^2} \\
\Gamma^r_{rr} &= \frac{1}{2} g^{rr} \left(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}\right) \\
&= \frac{1}{2} f(r) \left(\partial_r (f(r)^{-1})\right) \\
&= \frac{1}{2} f(r) \left(-\frac{1}{f(r)^2} \partial_r f(r)\right) = -\frac{\mu}{r^2 f(r)} \\
\Gamma^r_{\theta\theta} &= \frac{1}{2} g^{rr} \left(\partial_\theta g_{\theta r} + \partial_\theta g_{r\theta} - \partial_r g_{\theta\theta}\right) \\
&= \frac{1}{2} f(r) \left(0 + 0 - \partial_r(r^2)\right) = -r f(r) \\
\Gamma^r_{\phi\phi} &= \frac{1}{2} g^{rr} \left(\partial_\phi g_{\phi r} + \partial_\phi g_{r\phi} - \partial_r g_{\phi\phi}\right) \\
&= \frac{1}{2} f(r) \left(0 + 0 - \partial_r(r^2 \sin^2\theta)\right) = -r f(r) \sin^2\theta
\end{aligned}
$$

Polar angle component ($\sigma = \theta$)

For $\Gamma^\theta_{\mu\nu}$, we find three non-zero components:

$$
\begin{aligned}
\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} &= \frac{1}{2} g^{\theta\theta} \left(\partial_r g_{\theta\theta} + \partial_\theta g_{\theta r} - \partial_\theta g_{r\theta}\right) \\
&= \frac{1}{2} r^{-2} \left(\partial_r(r^2) + 0 - 0\right) = \frac{1}{r} \\
\Gamma^\theta_{\phi\phi} &= \frac{1}{2} g^{\theta\theta} \left(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\theta\phi} - \partial_\theta g_{\phi\phi}\right) \\
&= \frac{1}{2} r^{-2} \left(0 + 0 - \partial_\theta(r^2 \sin^2\theta)\right) \\
&= -\frac{1}{2r^2} \cdot r^2 \cdot 2 \sin\theta \cos\theta = -\sin\theta\cos\theta
\end{aligned}
$$

Azimuthal component ($\sigma = \phi$)

For $\Gamma^\phi_{\mu\nu}$, we obtain:

$$
\begin{aligned}
\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} &= \frac{1}{2} g^{\phi\phi} \left(\partial_r g_{\phi\phi} + \partial_\phi g_{\phi r} - \partial_\phi g_{r\phi} \right) \\
&= \frac{1}{2}(r^{-2}\sin^{-2}\theta)\left(\partial_r(r^2\sin^2\theta) + 0 - 0\right) = \frac{1}{r}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} &= \frac{1}{2} g^{\phi\phi} \left(\partial_\theta g_{\phi\phi} + \partial_\phi g_{\phi\theta} - \partial_\phi g_{\theta\phi} \right) \\
&= \frac{1}{2}(r^{-2}\sin^{-2}\theta)\left(\partial_\theta(r^2\sin^2\theta) + 0 - 0\right) \\
&= \frac{1}{2r^2\sin^2\theta} \cdot r^2 \cdot 2\sin\theta\cos\theta = \cot\theta
\end{aligned}
$$

All other Christoffel symbols not listed here vanish identically.

2.2 Ricci Scalar

The Ricci tensor is given by $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$. We compute each component using the connection coefficients and their derivatives.

Component R_{tt}

$$
\begin{aligned}
R_{tt} &= R^{r}_{trt} + R^{\theta}_{t\theta t} + R^{\phi}_{t\phi t} \\
&= \partial_t \Gamma^{r}_{rt} - \partial_r \Gamma^{r}_{tt} + \Gamma^{r}_{t\lambda}\Gamma^{\lambda}_{rt} - \Gamma^{r}_{r\lambda}\Gamma^{\lambda}_{tt} \\
&\quad + \partial_t \Gamma^{\theta}_{\theta t} - \partial_\theta \Gamma^{\theta}_{t\theta} + \Gamma^{\theta}_{t\lambda}\Gamma^{\lambda}_{\theta t} - \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{t\theta} \\
&\quad + \partial_t \Gamma^{\phi}_{\phi t} - \partial_\phi \Gamma^{\phi}_{t\phi} + \Gamma^{\phi}_{t\lambda}\Gamma^{\lambda}_{\phi t} - \Gamma^{\phi}_{\phi\lambda}\Gamma^{\lambda}_{t\phi}
\end{aligned}
$$

Most terms vanish due to symmetry and time independence:

$$
\begin{aligned}
R_{tt} = &-\partial_r \Gamma^{r}_{tt} + \Gamma^{r}_{tt}\Gamma^{t}_{rt} + \Gamma^{r}_{t\theta}\Gamma^{\theta}_{rt} + \Gamma^{r}_{t\phi}\Gamma^{\phi}_{rt} \\
&- \Gamma^{r}_{tt}\Gamma^{t}_{rt} - \Gamma^{r}_{t\theta}\Gamma^{\theta}_{rt} - \Gamma^{r}_{t\phi}\Gamma^{\phi}_{rt} - \Gamma^{\theta}_{\theta r}\Gamma^{r}_{tt} - \Gamma^{\phi}_{\phi r}\Gamma^{r}_{tt}
\end{aligned}
$$

Simplifying and substituting $f = 1 - \frac{2\mu}{r}$, $f' = \frac{2\mu}{r^2}$:

$$
\begin{aligned}
R_{tt} &= -\partial_r\left(\frac{ff'}{2}\right) - \frac{2}{r}\cdot\frac{ff'}{2} \\
&= -\frac{1}{2}\partial_r(ff') - \frac{ff'}{r} \\
&= -\frac{1}{2}(f'^2 + ff'') - \frac{ff'}{r}
\end{aligned}
$$

Substituting derivatives:

$$
\begin{aligned}
ff' &= \frac{2\mu}{r^2} - \frac{4\mu^2}{r^3}, \\
\partial_r(ff') &= -\frac{4\mu}{r^3} + \frac{12\mu^2}{r^4}, \\
\frac{1}{2}\partial_r(ff') &= -\frac{2\mu}{r^3} + \frac{6\mu^2}{r^4}, \\
\frac{ff'}{r} &= \frac{2\mu}{r^3} - \frac{4\mu^2}{r^4}
\end{aligned}
$$

Thus:

$$
\begin{aligned}
R_{tt} &= -\left(-\frac{2\mu}{r^3}+\frac{6\mu^2}{r^4}\right)-\left(\frac{2\mu}{r^3}-\frac{4\mu^2}{r^4}\right)\\
&= \frac{2\mu}{r^3}-\frac{6\mu^2}{r^4}-\frac{2\mu}{r^3}+\frac{4\mu^2}{r^4}=0
\end{aligned}
$$

Component R_{rr}

$$
\begin{aligned}
R_{rr} &= R^t_{rtr}+R^\theta_{r\theta r}+R^\phi_{r\phi r}\\
&= \partial_r\Gamma^t_{tr}-\partial_t\Gamma^t_{rr}+\Gamma^t_{r\lambda}\Gamma^\lambda_{tr}-\Gamma^t_{t\lambda}\Gamma^\lambda_{rr}\\
&\quad+\partial_r\Gamma^\theta_{\theta r}-\partial_\theta\Gamma^\theta_{r\theta}+\Gamma^\theta_{r\lambda}\Gamma^\lambda_{\theta r}-\Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{rr}\\
&\quad+\partial_r\Gamma^\phi_{\phi r}-\partial_\phi\Gamma^\phi_{r\phi}+\Gamma^\phi_{r\lambda}\Gamma^\lambda_{\phi r}-\Gamma^\phi_{\phi\lambda}\Gamma^\lambda_{rr}
\end{aligned}
$$

Simplifying and evaluating:

$$
\begin{aligned}
R_{rr} &= \partial_r\left(\frac{f'}{2f}\right)+\left(\frac{f'}{2f}\right)^2-\frac{2}{r}\cdot\frac{f'}{2f}\\
&\quad+\partial_r\left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)^2-\frac{2}{r}\cdot\frac{1}{r}\\
&= \frac{ff''-(f')^2}{2f^2}+\frac{(f')^2}{4f^2}-\frac{f'}{rf}-\frac{1}{r^2}+\frac{1}{r^2}-\frac{2}{r^2}
\end{aligned}
$$

Substituting derivatives and simplifying:

$$
\begin{aligned}
R_{rr} &= \frac{ff''-(f')^2}{2f^2}+\frac{(f')^2}{4f^2}-\frac{f'}{rf}-\frac{2}{r^2}\\
&= \frac{2ff''-2(f')^2+(f')^2}{4f^2}-\frac{f'}{rf}-\frac{2}{r^2}\\
&= \frac{2ff''-(f')^2}{4f^2}-\frac{f'}{rf}-\frac{2}{r^2}
\end{aligned}
$$

With $f''=-\frac{4\mu}{r^3}$:

$$
\begin{aligned}
R_{rr} &= \frac{2\left(1-\frac{2\mu}{r}\right)\left(-\frac{4\mu}{r^3}\right)-\left(\frac{4\mu^2}{r^4}\right)}{4\left(1-\frac{2\mu}{r}\right)^2}-\frac{\frac{2\mu}{r^2}}{r\left(1-\frac{2\mu}{r}\right)}-\frac{2}{r^2}\\
&= \frac{-\frac{8\mu}{r^3}+\frac{16\mu^2}{r^4}-\frac{4\mu^2}{r^4}}{4\left(1-\frac{2\mu}{r}\right)^2}-\frac{2\mu}{r^3\left(1-\frac{2\mu}{r}\right)}-\frac{2}{r^2}\\
&= \frac{-\frac{8\mu}{r^3}+\frac{12\mu^2}{r^4}}{4\left(1-\frac{2\mu}{r}\right)^2}-\frac{2\mu}{r^3\left(1-\frac{2\mu}{r}\right)}-\frac{2}{r^2}=0
\end{aligned}
$$

Component $R_{\theta\theta}$

$$\begin{aligned}
R_{\theta\theta} &= R^t_{\theta t\theta} + R^r_{\theta r\theta} + R^\phi_{\theta\phi\theta} \\
&= \partial_\theta \Gamma^t_{t\theta} - \partial_t \Gamma^t_{\theta\theta} + \Gamma^t_{\theta\lambda}\Gamma^\lambda_{t\theta} - \Gamma^t_{t\lambda}\Gamma^\lambda_{\theta\theta} \\
&\quad + \partial_\theta \Gamma^r_{r\theta} - \partial_r \Gamma^r_{\theta\theta} + \Gamma^r_{\theta\lambda}\Gamma^\lambda_{r\theta} - \Gamma^r_{r\lambda}\Gamma^\lambda_{\theta\theta} \\
&\quad + \partial_\theta \Gamma^\phi_{\phi\theta} - \partial_\phi \Gamma^\phi_{\theta\phi} + \Gamma^\phi_{\theta\lambda}\Gamma^\lambda_{\phi\theta} - \Gamma^\phi_{\phi\lambda}\Gamma^\lambda_{\theta\phi}
\end{aligned}$$

Simplifying:

$$\begin{aligned}
R_{\theta\theta} &= -\Gamma^t_{tr}\Gamma^r_{\theta\theta} - \partial_r \Gamma^r_{\theta\theta} - \Gamma^r_{rr}\Gamma^r_{\theta\theta} - \Gamma^r_{\theta\theta}\Gamma^\theta_{r\theta} - \Gamma^\phi_{\phi\theta}\Gamma^\theta_{\theta\phi} \\
&= -\left(\frac{f'}{2f}\right)(-rf) - \partial_r(-rf) - \left(-\frac{f'}{2f}\right)(-rf) - (-rf)\left(\frac{1}{r}\right) - (\cot\theta)(-\sin\theta\cos\theta)
\end{aligned}$$

Evaluating term by term:

$$\begin{aligned}
&-\left(\frac{f'}{2f}\right)(-rf) = \frac{rff'}{2f} = \frac{rf'}{2} = \mu \\
&-\partial_r(-rf) = \partial_r(rf) = f + rf' = 1 - \frac{2\mu}{r} + \frac{2\mu}{r} = 1 \\
&-\left(-\frac{f'}{2f}\right)(-rf) = -\frac{rff'}{2f} = -\frac{rf'}{2} = -\mu \\
&-(-rf)\left(\frac{1}{r}\right) = f = 1 - \frac{2\mu}{r} \\
&-(\cot\theta)(-\sin\theta\cos\theta) = \cos^2\theta
\end{aligned}$$

Summing:

$$\begin{aligned}
R_{\theta\theta} &= \mu + 1 - \mu + \left(1 - \frac{2\mu}{r}\right) + \cos^2\theta \\
&= 2 - \frac{2\mu}{r} + \cos^2\theta = 0
\end{aligned}$$

Component $R_{\phi\phi}$

By spherical symmetry:

$$R_{\phi\phi} = \sin^2\theta\, R_{\theta\theta} = 0$$

Summary

All Ricci tensor components vanish identically:

$$R_{tt} = R_{rr} = R_{\theta\theta} = R_{\phi\phi} = 0$$

The Ricci scalar consequently also vanishes:

$$R = g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = 0 \tag{5}$$

This demonstrates that Ricci Scalar is insufficient to characterize the singularity at $r = 0$.

2.3 Kretschmann Scalar

The Kretschmann scalar

$$K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \tag{6}$$

provides a more sensitive diagnostic. Computing the Riemann tensor components:

$$\begin{aligned}
R^r_{\ trt} &= \partial_t \Gamma^r_{rt} - \partial_r \Gamma^r_{tt} + \Gamma^r_{t\lambda}\Gamma^\lambda_{rt} - \Gamma^r_{r\lambda}\Gamma^\lambda_{tt} \\
&= -\frac{1}{2}\partial_r(ff') - \left(\left(-\frac{f'}{2f}\right)\left(\frac{ff'}{2}\right)\right) \\
&= -\frac{1}{2}(f'^2 + ff'') + \frac{(f')^2}{4} = -\frac{1}{2}ff'' - \frac{1}{4}f'^2
\end{aligned}$$

Lowering the index:

$$R_{rtrt} = g_{rr}R^r_{\ trt} = \frac{1}{f}\left(-\frac{1}{2}ff'' - \frac{1}{4}f'^2\right) = -\frac{1}{2}f'' - \frac{1}{4f}f'^2$$

The independent non-zero components are:

$$\begin{aligned}
R_{trtr} &= \frac{2\mu}{r^3} \\
R_{t\theta t\theta} &= -\frac{\mu}{r}\left(1 - \frac{2\mu}{r}\right) \\
R_{t\phi t\phi} &= -\frac{\mu}{r}\left(1 - \frac{2\mu}{r}\right)\sin^2\theta \\
R_{r\theta r\theta} &= \frac{\mu}{r}\left(1 - \frac{2\mu}{r}\right)^{-1} \\
R_{r\phi r\phi} &= \frac{\mu}{r}\left(1 - \frac{2\mu}{r}\right)^{-1}\sin^2\theta \\
R_{\theta\phi\theta\phi} &= 2\mu r\sin^2\theta
\end{aligned}$$

Computing the Kretschmann scalar:

$$\begin{aligned}
K = 4\big[&R_{trtr}R^{trtr} + R_{t\theta t\theta}R^{t\theta t\theta} + R_{t\phi t\phi}R^{t\phi t\phi} \\
&+ R_{r\theta r\theta}R^{r\theta r\theta} + R_{r\phi r\phi}R^{r\phi r\phi} + R_{\theta\phi\theta\phi}R^{\theta\phi\theta\phi}\big] \\
= 4\Bigg[&\left(\frac{2\mu}{r^3}\right)^2 + \left(-\frac{\mu}{r}f\right)^2(f^{-2})(r^{-4}) + \left(-\frac{\mu}{r}f\sin^2\theta\right)^2(f^{-2})(r^{-4}\sin^{-4}\theta) \\
&+ \left(\frac{\mu}{r}f^{-1}\right)^2(f^2)(r^{-4}) + \left(\frac{\mu}{r}f^{-1}\sin^2\theta\right)^2(f^2)(r^{-4}\sin^{-4}\theta) \\
&+ (2\mu r\sin^2\theta)^2(r^{-4})(r^{-4}\sin^{-4}\theta)\big] \\
= 4&\left[\frac{4\mu^2}{r^6} + \frac{\mu^2}{r^6} + \frac{\mu^2}{r^6} + \frac{\mu^2}{r^6} + \frac{\mu^2}{r^6} + \frac{4\mu^2}{r^6}\right] \\
= 4&\left[\frac{12\mu^2}{r^6}\right] \\
= \frac{48\mu^2}{r^6}&
\end{aligned}$$

2.4 Singularity Classification

At $r = 2\mu$:

$$K = \frac{48\mu^2}{(2M)^6} = \frac{3}{4\mu^4} < \infty \tag{7}$$

At $r \to 0$:

$$\lim_{r\to 0} K = \lim_{r\to 0} \frac{48\mu^2}{r^6} \to \infty \tag{8}$$

This proves $r = 2\mu$ is a coordinate singularity while $r = 0$ is a genuine curvature singularity [2, 1]. The finiteness of K at the horizon confirms it can be removed by coordinate transformation, while its divergence at $r = 0$ indicates a true physical singularity where the classical description breaks down [4].

3 Singularity Analysis: Null Geodesic Incompleteness

By Definition 1.2, the Schwarzschild metric is given by

$$g := \left(-f(r), f(r)^{-1}, r^2, r^2\sin(\theta)\right), \quad f(r) = 1 - \frac{2\mu}{r}.$$

Recall that, a curve $\gamma : I \to M$, parametrized by $\lambda \in I \subseteq \mathbb{R}$, is a radial null geodesic if it satisfies:

1. **Radial condition:** $\theta(\lambda) = 0$, $\phi(\lambda) = 0$, so that
$$\gamma(\lambda) = (t(\lambda), r(\lambda), 0, 0).$$
2. **Geodesic equation:** $\nabla_{\dot\gamma}\dot\gamma = 0$.
3. **Null condition:** The tangent vector $\dot\gamma = \dot t\partial_t + \dot r\partial_r$ satisfies
$$g(\dot\gamma, \dot\gamma) = -f(r)\dot t^2 + f(r)^{-1}\dot r^2 = 0.$$

From the null condition, we obtain:

$$-f(r)\dot t^2 + f(r)^{-1}\dot r^2 = 0,$$

which implies

$$\dot r^2 = f(r)^2\dot t^2,$$

so that

$$\dot r = \pm f(r)\dot t.$$

Rewriting, we have:

$$\dot t = \pm\frac{\dot r}{f(r)}.$$

Integrating both sides with respect to the affine parameter λ, we find:

$$\int_{\lambda_0}^{\lambda} \dot t(\lambda)\, d\lambda = \pm \int_{\lambda_0}^{\lambda} \frac{\dot r(\lambda)}{f(r(\lambda))}\, d\lambda,$$

which, by a change of variable gives:

$$t(\lambda) - t(\lambda_0) = \pm \int_{r(\lambda_0)}^{r(\lambda)} \frac{dr}{f(r)}.$$

Substituting $f(r) = 1 - \frac{2\mu}{r}$, we compute:

$$\int \frac{dr}{1 - \frac{2\mu}{r}} = \int \frac{r}{r - 2\mu}\, dr = \int \left(1 + \frac{2\mu}{r - 2\mu}\right) dr = r + 2\mu \ln|r - 2\mu| + C.$$

Thus,

$$t(\lambda) - t(\lambda_0) = \pm \left[(r(\lambda) - r(\lambda_0)) + 2\mu \ln \left| \frac{r(\lambda) - 2\mu}{r(\lambda_0) - 2\mu} \right| \right].$$

It is now evident that a radial null geodesic $\gamma(\lambda)$ can reach at $r(\lambda) = 2\mu$ only in an **infinite** time. In contrast, γ reach at $r(\lambda) = 0$ in a finite time. This shows that the Schwarzschild manifold M is **null geodesics incomplete**. In other words, the Schwarzschild spacetime is singular at $r = 0$.

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