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The anatomy of non-vacuum Kasner branes

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Abstract: This report is based on the work done for the research internship on “time dependent backgrounds in string theory” during July to October 2025. We review “Kasner branes with arbitrary signature” by Sabra [1] and provide through calculation using the method he presented. We find time dependent solutions for the theory of gravity with dilaton field and form flux. Moreover, we also investigate beyond Kasner-like solutions. This work has significance in the field of supergravity.

Approval

The internship report titled "**The anatomy of non-vacuum Kasner branes**" submitted by **Walid Hasan**, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfilment of the requirements of the internship program.

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Contents

1 Introduction

Kasner metric provides an exact vacuum solutions for anisotropic and spatially homogeneous cosmologies in general relativity. In some cosmological models, near the big-bang singularity the spatial points decouple and evolution of each point can be described by Kasner epoch. Hence, they provide a canonical local description of geometry near cosmological singularities. Kasner metric is used to model expansion or contraction of universes at different rate along different axes. In (3+1) dimensions, Kasner metric describes a homogenous universe where two directions expands while one contracts among the spatial axes as $t \to 0$. However, further analysis shows due to having three dimensional Ricci tensor makes the Kasner solution unstable near singularity, as shown by Belinski and Henneaux in [2]. We define cosmological singularity as a spacelike, time-ending (or time-beginning) singularity in which curvature invariants and invariant matter fields diverge on a three-dimensional manifold. [2] Kasner metric falls under Bianchi type I classification. However, it also acts as the asymptotic building block for more curved universes that fall under Bianchi type VII or IX. In these types of universes, Kasner metric manifests different scenarios such as gravitational turbulence and chaotic epoch near singularities.

We get the idea of spacetime being fundamentally higher dimensional from string theory. [3] In realistic high-energy settings motivated by string/M-theory, the early universe is rarely vacuum. The diaton scalar moduli and antisymmetric tensor fluxes are natural degrees of freedom in this scenario. Time dependent solutions are important in M-theory as the dualities act non-trivially on dynamics. When the dilaton and form flux are time dependent, they allow us to modify dualities from formal symmetries of stationary vacua into nontrivial, solution-generating constraints on dynamics. An evolving background expose how type II/heretoric strings and M-theory can give type IIA*, IIB*, $M*$ (9+2) and $M^0(6+5)$ theories.

[4] Such solutions allow us to investigate holography, dual symmetries, AdS/CFT correspondence, and cosmological singularities and instabilities.

In [1], Sabra focuses on non-vacuum Kasner-like solutions supported by a dilaton and p-form flux. He proposes a model gravitation theory with arbitary rank form field coupled with dilaton scalar and presents a method to find static and time dependent solutions. This setup serves as a toy model for string theory: the dilaton scalar and the form field strength stand in for the familiar NS–NS/R–R sectors of supergravity.

In this report, we have attempted to reproduce the method introduced in [1] and find the general Kasner-like brane solutions for three cases. Firstly, we considered a gravity theory with a form field. Then we took a gravity theory with an scalar field. Lastly, we used the model theory with gravity, form field and coupled dilaton scalar. Moreover, we have reviewed classical field theory and Einstein Hilbert action in Appendix A and a brief overview of differential forms in Appendix B.

2 Kasner Metric

In 3+1 dimension, we can write the metric

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + t^{2p}\,\mathrm{d}x^2 + t^{2q}\,\mathrm{d}y^2 + t^{2r}\,\mathrm{d}z^2 \tag{2.1}$$

Here

$$g_{\mu\nu} = \mathrm{diag}(-1, t^{2p}, t^{2q}, t^{2r}) \quad \& \quad g^{\mu\nu} = \mathrm{diag}(-1, t^{-2p}, t^{-2q}, t^{-2r}) \tag{2.2}$$

Christoffel symbols defined as

$$\Gamma_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{t\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \tag{2.3}$$

Non-zero christoffel symbols has lower indices $\mu\nu = xx, yy, zz$ when upper index $\rho = t$. And when the upper indices are either x, y, z, then lower indices are tx, ty, and tz respectively.

$$\begin{aligned}
\Gamma_{xx}{}^{t} &= \frac{1}{2} g^{t\lambda}(\partial_x g_{x\lambda} + \partial_x g_{x\lambda} - \partial_\lambda g_{xx}) \\
&= \frac{1}{2} g^{tt}(\partial_x g_{xt} + \partial_x g_{xt} - \partial_t g_{xx}) \\
&= \frac{1}{2}(-1)(-2pt^{2p-1}) = pt^{2p-1}
\end{aligned} \tag{2.4}$$

Similarly, $\Gamma_{yy}{}^{t} = qt^{2q-1}$ and $\Gamma_{zz}{}^{t} = rt^{2r-1}$.

$$\begin{aligned}
\Gamma_{tx}{}^{x} &= \frac{1}{2} g^{xx}(\partial_t g_{xx} + \partial_x g_{tx} - \partial_x g_{tx}) \\
&= \frac{1}{2} t^{-2p}(2pt^{2p-1}) \\
&= pt^{-1}
\end{aligned} \tag{2.5}$$

Similarly, $\Gamma_{ty}{}^{y} = qt^{-1}$ and $\Gamma_{tz}{}^{z} = rt^{-1}$.

Now the Ricci tensor is defined as

$$R_{\mu\nu} = \partial_\sigma \Gamma_{\mu\nu}{}^{\sigma} + \Gamma_{\kappa\sigma}{}^{\sigma}\Gamma_{\mu\nu}{}^{\kappa} - \partial_\mu \Gamma_{\mu\sigma}{}^{\sigma} - \Gamma_{\kappa\nu}{}^{\sigma}\Gamma_{\mu\sigma}{}^{\kappa} \tag{2.6}$$

The non zero components are R_{tt}, R_{xx}, R_{yy} and R_{zz}.

- R_{tt}

$$\begin{aligned}
R_{tt} &= -\partial_t \Gamma_{t\sigma}{}^{\sigma} - \Gamma_{\kappa t}{}^{\sigma}\Gamma_{t\sigma}{}^{\kappa} \\
&= -\partial_t\left(\Gamma_{tx}{}^{x} + \Gamma_{ty}{}^{y} + \Gamma_{tz}{}^{z}\right) - \left(\Gamma_{ti}{}^{i}\right)^2 \\
&= -\frac{p+q+r}{t^2} - \frac{p^2+q^2+r^2}{t^2} \\
&= -\frac{1}{t^2}(p+q+r-p^2-q^2-r^2)
\end{aligned} \tag{2.7}$$

- R_{xx}

$$\begin{aligned} R_{xx} &= \partial_\sigma \Gamma_{xx}{}^\sigma + \Gamma_{\kappa\sigma}{}^\sigma \Gamma_{xx}{}^\kappa - \partial_x \Gamma_{x\sigma}{}^\sigma - \Gamma_{\kappa x}{}^\sigma \Gamma_{x\sigma}{}^\kappa \\ &= \partial_t \Gamma_{xx}{}^t + \Gamma_{ti}{}^i \Gamma_{xx}{}^t - 0 - 2\Gamma_{tx}{}^x \Gamma_{xx}{}^t \\ &= p(2p-1)t^{2p-2} + pt^{2p-2}(p+q+r) - 2p^2 t^{2p-2} \\ &= p(p+q+r-1)t^{2p-2} \end{aligned} \tag{2.8}$$

Similarly,

$$R_{yy} = q(p+q+r-1)t^{2q-2} \quad \& \quad R_{zz} = r(p+q+r-1)t^{2r-2} \tag{2.9}$$

Now, if we have $R_{\mu\nu} = 0$, then

$$p+q+r = p^2+q^2+r^2 = 1 \tag{2.10}$$

Hence, Kasner metric satisfies the Einstein field equations in vaccum.

2.1 General form of Kasner Metric

We can generalize the Kasner metric to arbitrary space-time dimensions and signatures. In d dimensions

$$\mathrm{d}s^2 = \epsilon_0 \, \mathrm{d}\tau^2 + \sum_{i=1}^{d-1} \epsilon_i \tau^{2a_i} \, \mathrm{d}x_i{}^2 \tag{2.1.1}$$

with conditions

$$\sum_i^{d-1} a_i = \sum_i^{d-1} a_i^2 = 1 \tag{2.1.2}$$

We can find the Ricci tensors

$$\begin{aligned} R_{\tau\tau} &= \frac{1}{\tau^2} \sum_{i=1}^{d-1} (a_i - a_i^2) \\ R_{x_i x_i} &= \epsilon_0 \epsilon_i \tau^{-2a_i - 2} a_i \left(1 - \sum_{j=1}^{d-1} a_j \right) \end{aligned} \tag{2.1.3}$$

We get the vacuum solutions if we impose the Kasner conditions.

3 The usual suspects: three actions in [1]

3.1 d-dimension gravity theory with m-form F_m

Action

$$S = \int \mathrm{d}^d x \sqrt{|g|} \left(R - \frac{\epsilon}{2m!} F_m^2 \right) \tag{3.1.1}$$

In forms notation

$$S = \int_M \left(R \star 1 - \frac{\epsilon}{2} F \wedge \star F \right) \tag{3.1.2}$$

We define the potential A.

$$F_m = \mathrm{d}A_{m-1} \Rightarrow \delta F = \mathrm{d}\, \delta A \tag{3.1.3}$$

First we vary the action with respect to A

$$\begin{aligned} \delta_A S &= -\frac{\epsilon}{2} \int \delta F \wedge \star F \\ &= -\frac{\epsilon}{2} \int \delta A \wedge (-1)^p \, \mathrm{d} \star F \end{aligned} \tag{3.1.4}$$

Setting $\delta_A S = 0$,

$$\begin{aligned} &\mathrm{d} \star F = 0 \\ \Rightarrow\ &\nabla_\mu F^{\mu v_2 \dots v_m} = 0 \\ \Rightarrow\ &\partial_\mu \left(\sqrt{|g|} F^{\mu v_2 \dots v_m} \right) = 0 \end{aligned} \tag{3.1.5}$$

This is the second equation of motion in Equation 1.11 in [1].

Now we vary the action with respect to the metric g. The action

$$S = S_{EH} + S_F \tag{3.1.6}$$

Then

$$\delta S_{EH} = \int \mathrm{d}^d x \sqrt{|g|} \Big(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Big) \delta g^{\mu\nu} \tag{3.1.7}$$

We define $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. Now

$$\delta S_F = -\frac{\epsilon}{2m!} \int \mathrm{d}^d x \big(\delta\sqrt{|g|} F^2 + \sqrt{|g|} \delta F^2 \big) \tag{3.1.8}$$

In Equation A.3.5 we shown,

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|} g_{\mu\nu} \delta g^{\mu\nu} \tag{3.1.9}$$

and

$$F^2 = F_{\mu_1\mu_2\cdots\mu_m} \delta F^{\mu_1\cdots\mu_m} \tag{3.1.10}$$

We write

$$\begin{aligned} \delta F^{\mu_1\mu_2\cdots\mu_m} &= \sum_{k=1}^{m} \Bigg(\prod_{j\neq k} g^{\mu_j\nu_j} \Bigg) \delta g^{\mu_k\nu_k} F_{\nu_1\ldots\nu_m} \\ \delta F^2 &= \sum_{k=1}^{m} F_{\mu_1\mu_2\cdots\mu_m} \prod_{j\neq k} g^{\mu_j\nu_j} \delta g^{\mu_k\nu_k} F_{\nu_1\ldots\nu_m} \end{aligned} \tag{3.1.11}$$

The first term $k = 1$

$$\begin{aligned} F_{\mu_1\mu_2\cdots\mu_m} g^{\mu_2\nu_2} \ldots g^{\mu_m\nu_m} \delta g^{\mu_1\nu_1} F_{\nu_1\ldots\nu_m} &= \delta g^{\mu_1\nu_1} F_{\mu_1\ldots\mu_m} F_{\nu_1}{}^{\mu_2\cdots\mu_m} \\ &= \delta g^{\mu\nu} F_{\mu\alpha_2\ldots\alpha_m} F_{\nu}{}^{\alpha_2\ldots\alpha_m} \end{aligned} \tag{3.1.12}$$

Since there are m similar terms, then

$$\delta F^2 = m F_{\mu\alpha_2\ldots\alpha_m} F_{\nu}{}^{\alpha_2\ldots\alpha_m} \delta g^{\mu\nu} \tag{3.1.13}$$

Hence

$$\delta S_F = -\frac{\epsilon}{2m!} \int \mathrm{d}^d x \sqrt{|g|} \Big[-\frac{1}{2} g_{\mu\nu} F^2 + m F_{\mu\alpha_2\ldots\alpha_m} F_{\nu}{}^{\alpha_2\ldots\alpha_m} \Big] \delta g^{\mu\nu} \tag{3.1.14}$$

We know stress energy tensor

$$\begin{aligned} T_{\mu\nu} &\equiv -\frac{1}{\sqrt{|g|}} \frac{\delta S_F}{\delta g^{\mu\nu}} \\ &= \frac{\epsilon}{2} \Bigg[\frac{1}{(m-1)!} F_{\mu\alpha_2\ldots\alpha_m} F_{\nu}{}^{\alpha_2\ldots\alpha_m} - \frac{1}{2m!} g_{\mu\nu} F^2 \Bigg] \end{aligned} \tag{3.1.15}$$

Taking the trace

$$\begin{aligned} T = g^{\mu\nu} T_{\mu\nu} &= \frac{\epsilon}{2} \Bigg[\frac{1}{(m-1)!} F_{\mu\alpha_2\ldots\alpha_m} F^{\mu\alpha_2\ldots\alpha_m} - \frac{1}{2m!} g^{\mu\nu} g_{\mu\nu} F^2 \Bigg] \\ &= \frac{\epsilon}{2} \Bigg[\frac{1}{(m-1)!} F^2 - \frac{d}{2m!} F^2 \Bigg] \\ &= \frac{\epsilon}{2} \Bigg(\frac{2m-d}{2m!} F^2 \Bigg) \end{aligned} \tag{3.1.16}$$

We know the traced reverse Einstein equation

$$\begin{aligned} R_{\mu\nu} &= T_{\mu\nu} - \frac{1}{d-2} g_{\mu\nu} T \\ &= \epsilon \Bigg[\frac{1}{2(m-1)!} F_{\mu\alpha_2\ldots\alpha_m} F_{\nu}{}^{\alpha_2\ldots\alpha_m} - g_{\mu\nu} F^2 \frac{1}{d-2} \Bigg(\frac{1}{4m!} + \frac{2m-d}{4m!} \Bigg) \Bigg] \\ &= \epsilon \Bigg[\frac{1}{2(m-1)!} F_{\mu\alpha_2\ldots\alpha_m} F_{\nu}{}^{\alpha_2\ldots\alpha_m} - g_{\mu\nu} F^2 \frac{m-1}{2m!(d-2)} \Bigg] \end{aligned} \tag{3.1.17}$$

This is the first equation of motion in Equation 1.11 from [1].

3.2 Kasner backgrounds for this action

We use Equation (2.1.1) as the metric ansatz and we take m-form to be

$$F_m = P\,\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_m\,. \tag{3.2.1}$$

Here $F_{x_i \dots x_m} = P$. Then

$$\begin{aligned} F^2 &= m! g^{x_1 x_1} \dots g^{x_m x_m} F^2_{x_1 \dots x_m} \\ &= m! \prod_{i=1}^{m} \epsilon_i \tau^{-2a_i} P^2 \\ &= m! E \tau^{-2\sum_i^m a_i} P^2 \end{aligned} \tag{3.2.2}$$

with $E = \epsilon_1 \dots \epsilon_m$. And for $i \leq m$

$$\begin{aligned} F_{x_i a_2 \dots a_m} F_{x_i}^{\,a_2 \dots a_m} &= (m-1)! P g^{a_2 a_2} \dots g^{a_m a_m} F_{x_i a_2 \dots a_m} \\ &= (m-1)! p^2 \prod_{k=1, k\neq i}^{m} \epsilon_k \tau^{-2a_k} \\ &= (m-1)! P^2 \frac{E}{\epsilon_i} \tau^{-2(\sum_k^m a_k - a_i)} \end{aligned} \tag{3.2.3}$$

The Einstein equations become for $\mu = \nu = \tau$

$$\begin{aligned} R_{\tau\tau} &= -\epsilon g_{\tau\tau} \frac{m-1}{2m!(d-2)} F^2 \\ \frac{1}{\tau^2} \sum_{i=1}^{d-1} (a_i - a_i^2) &= -\epsilon \epsilon_0 \frac{m-1}{2(d-2)} E P^2 \tau^{-2\sum_i^m a_i} \end{aligned} \tag{3.2.4}$$

Matching the power of τ

$$\sum_{i=1}^{m} a_i = 1 \tag{3.2.5}$$

Then we have

$$\sum_{i=1}^{d-1} (a_i - a_i^2) = -\epsilon \epsilon_0 E \frac{m-1}{2(d-2)} P^2 \tag{3.2.6}$$

Now, we consider the space component of Ricci tensor. The x_i indices run to $d-1$. Let consider $i \leq m$ case.

$$\begin{aligned} R_{x_i x_i} &= \epsilon \left(\frac{1}{2} P^2 \frac{E}{\epsilon_i} \tau^{-2(1-a_i)} - \frac{m-1}{2(d-2)} \epsilon_i E P^2 \tau^{2a_i - 2} \right) \\ \epsilon_0 a_i \left(1 - \sum_{j=1}^{d-1} a_j \right) &= \frac{\epsilon}{2} E P^2 \frac{d-m-1}{d-2} \tau^{2a_i - 2} \end{aligned} \tag{3.2.7}$$

The right hand side is independent of i. Hence, for $i \leq m$, we get all a_i to be equal.

$$\sum_{i=1}^{m} a_i = 1 \Rightarrow a_1 = a_2 = \dots = a_m = \frac{1}{m} = A \tag{3.2.8}$$

Again, if we consider $i \geq m$ case

$$\begin{aligned} R_{x_i x_i} &= -\frac{\epsilon \epsilon_i E}{2(d-2)} (m-1) P^2 \tau^{2a_i - 2} \\ \epsilon_0 a_i \left(1 - \sum_{j=1}^{d-1} a_j \right) &= -\frac{\epsilon}{2} E \frac{m-1}{d-2} P^2 \end{aligned} \tag{3.2.9}$$

Here, all a_i should be equal as well.

$$a_{m+1} = a_{m+2} = \dots = a_{d-1} = B \tag{3.2.10}$$

Then, the ratio of Equation (3.2.9) and Equation (3.2.7)

$$\begin{aligned} \frac{A}{B} &= -\frac{d-m-1}{m-1} \\ \Rightarrow B &= -\frac{m-1}{m(d-m-1)} \end{aligned} \tag{3.2.11}$$

Now, we have

$$\sum_{j=1}^{d-1} = \sum_{j=1}^{m} a_j + \sum_{j=m+1}^{d-1} a_j = m\frac{1}{m} - (d-1-m)\frac{m-1}{m(d-m-1)} = \frac{1}{m} \tag{3.2.12}$$

If we put the values into Equation (3.2.7)

$$\begin{aligned} \epsilon_0 \frac{1}{m(1-\frac{1}{m})} &= \frac{\epsilon}{2} E P^2 \frac{d-m-1}{d-2} \\ \Rightarrow P^2 &= 2\epsilon\epsilon_0\epsilon_1 \ldots \epsilon_m \frac{(m-1)(d-2)}{m^2(d-m-1)} \end{aligned} \tag{3.2.13}$$

Hence we found the solutions for the equations of motion.

3.3 d-dim gravity with a dynamic scalar field

We can write the action

$$S = \int \mathrm{d}^d x \sqrt{|g|} \Big(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \Big) \tag{3.3.1}$$

First we vary with respect to ϕ

$$\begin{aligned} \delta_\phi S &= -\int \mathrm{d}^d x \sqrt{|g|} \partial_\mu \phi \partial^\mu \, \delta\phi \\ &= -\int \mathrm{d}^d x \Big[\partial^\mu \Big(\sqrt{|g|} \partial_\mu \phi \delta\phi \Big) - \partial^\mu \Big(\sqrt{|g|} \partial_\mu \phi \Big) \delta\phi \Big] \\ &= \int \mathrm{d}^d x \; \partial_\mu \Big(\sqrt{|g|} \partial^\mu \phi \Big) \delta\phi \qquad \text{(dropping the boundary term)} \end{aligned} \tag{3.3.2}$$

Hence, extremizing the action

$$\partial_\mu \Big(\sqrt{|g|} \partial^\mu \phi \Big) = 0 \tag{3.3.3}$$

This is the equation of motion for scalar field.

Now, we vary with respect to the metric. The Einstein-Hilbert action part gives

$$\delta_g S_{EH} = \int \mathrm{d}^d x \Big(R - \frac{1}{2} g_{\mu\nu} R \Big) \delta g^{\mu\nu} \tag{3.3.4}$$

The scalar field part gives

$$\delta_g S_\phi = -\frac{1}{2} \int \mathrm{d}^d x \delta \Big(\sqrt{|g|} g^{\mu\nu} \Big) \partial_\mu \phi \partial_\nu \phi \tag{3.3.5}$$

We have

$$\delta \Big(\sqrt{|g|} g^{\mu\nu} \Big) = \sqrt{|g|} \Big(\delta g^{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\alpha\beta} \delta g^{\alpha\beta} \Big) \tag{3.3.6}$$

Then

$$\delta_g S_\phi = \int \mathrm{d}^d x \frac{\sqrt{|g|}}{2} \Big[-\partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g_{\mu\nu} \partial_\mu \phi \partial^\mu \phi \Big] \delta g^{\mu\nu} \tag{3.3.7}$$

Hence, the stress energy tensor

$$\begin{aligned} T_{\mu\nu} &= -\frac{1}{\sqrt{|g|}} \frac{\delta_g S_\phi}{\delta g^{\mu\nu}} \\ &= \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} g_{\mu\nu} \partial_\mu \phi \partial^\mu \phi \end{aligned} \tag{3.3.8}$$

Now trace

$$T = g^{\mu\nu}T_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{d}{4}\partial_\mu\phi\partial^\mu\phi = \frac{2-d}{4}\partial_\mu\phi\partial^\mu\phi \tag{3.3.9}$$

Trace reversed Einstein equation

$$\begin{aligned} R_{\mu\nu} &= T_{\mu\nu} - \frac{1}{d-2}g_{\mu\nu}T \\ &= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \left(\frac{1}{4} - \frac{d-2}{4(d-2)}\right)g_{\mu\nu}\partial_\mu\phi\partial^\mu\phi \\ &= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi \end{aligned} \tag{3.3.10}$$

This is the second equation of motion.

3.4 Kasner background for this action

We take Equation (2.1.1) as our metric ansatz. We have

$$|g| = |\epsilon_0\epsilon_1 \ldots \epsilon_{d-1}|\tau^{2\sum_i^{d-1} a_i} \Rightarrow \sqrt{|g|} = C\tau^{\sum_i^{d-1} a_i} \tag{3.4.1}$$

We have the Ricci tensors

$$R_{\tau\tau} = \frac{1}{2}\dot{\phi}^2 \tag{3.4.2}$$

and

$$R_{x_i x_i} = 0 \Rightarrow \sum_i^{d-1} a_i = 1 \tag{3.4.3}$$

This is one of the vacuum condition for Kasner metric. We consider $\phi = \phi(\tau)$, and write

$$\partial_\tau\left(\sqrt{|g|}\partial^\tau\varphi\right) = \partial_\tau(C\tau g^{\tau\tau}\partial_\tau\phi) = \epsilon_0 C\partial_\tau(\tau\partial_\tau\phi) = 0 \Rightarrow \tau\dot{\phi} = d_1 \tag{3.4.4}$$

$$\phi = d_1\ln\tau + d_2 \tag{3.4.5}$$

Now,

$$\begin{aligned} R_{\tau\tau} &= \frac{1}{\tau^2}\sum_i^{d-1}(a_i - a_i^2) = \frac{1}{2}\frac{d_1^2}{\tau^2} \\ &\Rightarrow \frac{d_1^2}{2} + \sum_i^{d-1} a_i^2 = 1 \end{aligned} \tag{3.4.6}$$

If $d_1 = 0$, then we would get both of the Kasner vacuum conditions. In that case, ϕ would become constant.

3.5 d-dim gravity with m-form field F_m and a dilaton scalar ϕ coupled to the field

The action

$$S = \int \mathrm{d}^d x\sqrt{|g|}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\epsilon}{2m!}e^{\beta\phi}F_m^2\right) \tag{3.5.1}$$

The coupled m-form field

$$\frac{1}{m!}F_m^2 = \frac{1}{m!}F_{\mu_1\cdots\mu_m}F^{\mu_1\cdots\mu_m} = F\wedge\star F \tag{3.5.2}$$

and

$$F_m = \mathrm{d}A_{m-1} \tag{3.5.3}$$

Then the action

$$S_F = -\frac{\epsilon}{2}\int e^{\beta\phi}F\wedge\star F \tag{3.5.4}$$

Varying the action gives

$$\begin{gathered} \mathrm{d}\left(e^{\beta\phi}\star F\right) = 0 \\ \partial_\mu\left(\sqrt{|g|}e^{\beta\phi}F^{\mu\nu_2\ldots\nu_m}\right) = 0 \end{gathered} \tag{3.5.5}$$

This is the second equation of motion in Eq 2.27 from [1].

Now we vary with respect the the scalar field. The action

$$S_\phi = -\frac{1}{2}\int \mathrm{d}^d s\sqrt{|g|}\partial_\mu\phi\partial^\mu\phi - \frac{\epsilon}{2m!}\int \mathrm{d}^d x\sqrt{|g|}e^{\beta\phi}F^2 \tag{3.5.6}$$

Varying the first part gives us

$$\delta_\phi S^1_\phi = \int \mathrm{d}^d x\sqrt{|g|}\partial_\mu\partial^\mu\phi\,\delta\phi \tag{3.5.7}$$

and the second part

$$\delta_\phi S^2_\phi = -\frac{\epsilon}{2m!}\int \mathrm{d}^d x\sqrt{|g|}\beta e^{\beta\phi}F^2\,\delta\phi \tag{3.5.8}$$

Hence, least action principle gives

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\partial^\mu\phi\right) = -\frac{\beta\epsilon}{2m!}e^{\beta\phi}F_{\mu_1\cdots\mu_m}F^{\mu_1\cdots\mu_m} \tag{3.5.9}$$

This is the third equation in Eq 2.27 from [1].

Now, we vary with respect to the metric. The Einstein Hilbert part gives

$$\delta_g S_{EH} = \int \mathrm{d}^d x\sqrt{|g|}\Big(R - \frac{1}{2}g_{\mu\nu}R\Big)\delta g^{\mu\nu} \tag{3.5.10}$$

the scalar field part gives

$$\delta_g S_\phi = \int \mathrm{d}^d x\frac{\sqrt{|g|}}{2}\Big[-\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g_{\mu\nu}\partial_\mu\phi\partial^\mu\phi\Big]\delta g^{\mu\nu} \tag{3.5.11}$$

then

$$R^\phi_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi \tag{3.5.12}$$

The coupled form part gives

$$\delta_g S_{F\phi} = -\frac{\epsilon}{2m!}\int \mathrm{d}^d x\sqrt{|g|}e^{\beta\phi}\Big[-\frac{1}{2}g_{\mu\nu}F^2 + mF_{\mu\alpha_2\ldots\alpha_m}F_\nu{}^{\alpha_2\ldots\alpha_m}\Big]\delta g^{\mu\nu} \tag{3.5.13}$$

then

$$R^F_{\mu\nu} = \epsilon e^{\beta\phi}\left[\frac{1}{2(m-1)!}F_{\mu\alpha_2\ldots\alpha_m}F_\nu{}^{\alpha_2\ldots\alpha_m} - g_{\mu\nu}F^2\frac{m-1}{2m!(d-2)}\right] \tag{3.5.14}$$

Hence, the ricci tensor

$$\begin{aligned} R_{\mu\nu} &= R^\phi_{\mu\nu} + R^F_{\mu\nu} \\ &= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{\epsilon e^{\beta\phi}}{2(m-1)!}\left[F_{\mu\alpha_2\ldots\alpha_m}F_\nu{}^{\alpha_2\ldots\alpha_m} - g_{\mu\nu}F^2\frac{m-1}{m(d-2)}\right] \end{aligned} \tag{3.5.15}$$

This is the first equation of motion in Eq 2.27 from [1].

4 Kasner-like brane solutions

Sabra proposes a generic metric solution

$$\mathrm{d}s^2 = e^{2U(\tau)}\left(\epsilon_0\,\mathrm{d}\tau^2 + \sum_{i=1}^{p}\epsilon_i\tau^{2a_i}\,\mathrm{d}x_i{}^2\right) + e^{2V(\tau)}\left(\sum_{j=p+1}^{d-1}\epsilon_j\tau^{2a_j}\,\mathrm{d}x_j{}^2\right) \tag{4.1}$$

here $\epsilon, \epsilon_i, \epsilon_j = \pm 1$ and a_i, a_j are constants. We have $d = p + q + 1$ and the Ricci tensors are

$$R_{\tau\tau} = -q\ddot{V} - p\ddot{U} - q\dot{V}\big(\dot{V} - \dot{U}\big) - \frac{1}{\tau}\big((s-l)\dot{U} + 2l\dot{V}\big) - \frac{1}{\tau^2}\sum_{k=1}^{d-1}(a_k^2 - a_k), \tag{4.2}$$

$$R_{x_i x_i} = -\epsilon_0\epsilon_i\tau^{2a_i}\left[\ddot{U} - \frac{a_i}{\tau^2} + \left(\dot{U} + \frac{a_i}{\tau}\right)\left((p-1)\dot{U} + q\dot{V} + \frac{l+s}{\tau}\right)\right], \tag{4.3}$$

and

$$R_{x_j x_j} = -\epsilon_0\epsilon_j e^{2V-2U}\tau^{2a_j}\left[\ddot{V} - \frac{a_j}{\tau^2} + \left(\dot{V} + \frac{a_j}{\tau}\right)\left(q\dot{V} + (p-1)\dot{U} + \frac{l+s}{\tau}\right)\right]. \tag{4.4}$$

We take the following definitions and assumptions.

$$l = \sum_{j=p+1}^{d-1} a_j, \quad s = \sum_{i=1}^{p} a_i \tag{4.5}$$

The Kasner condition implies $l + s = 1$. Moreover, we impose

$$V = -\frac{p-1}{q}U. \tag{4.6}$$

Now we can simplify the Ricci tensors.

$$R_{\tau\tau} = -\ddot{U} - \left[1 - 2\left(\frac{d-2}{q}\right)l\right]\frac{\dot{U}}{\tau} - \frac{(p-1)(d-2)}{q}\dot{U}^2, \tag{4.7}$$

$$R_{x_i x_i} = -\epsilon_0 \epsilon_i \tau^{2a_i}\left(\ddot{U} + \frac{\dot{U}}{\tau}\right), \tag{4.8}$$

$$R_{x_j x_j} = \varepsilon_0 \varepsilon_j e^{2\frac{2-d}{q}U}\tau^{2a_j}\frac{p-1}{q}\left(\ddot{U} + \frac{\dot{U}}{\tau}\right). \tag{4.9}$$

Now we take our form ansatz

$$F_p = P\, \mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_p \tag{4.10}$$

We compute

$$F^2 = p! P^2 (\epsilon_1 ... \epsilon_p) e^{-2pU}\tau^{-2s} \tag{4.11}$$

and

$$F_{x_i a_2 ... a_p} F_{x_i}^{\ a_2 ... a_p} = (p-1)! P^2 \frac{\epsilon_1 ... \epsilon_p}{\epsilon_i} e^{-2(p-1)U}\tau^{-2(s-a_i)} \tag{4.12}$$

4.1 Gravity theory with m-form

Our action in Equation (3.1.2) gives equations of motion Equation (3.1.5) and Equation (3.1.17). We write

$$\begin{gathered} R_{x_i x_i} = e^{-2(p-1)U}\tau^{-2(s-a_i)}\left[\frac{\epsilon\epsilon_i\epsilon_1\epsilon_p}{2}P^2\frac{q}{d-2}\right] \\ \Rightarrow \ddot{U} + \frac{\dot{U}}{\tau} = -\epsilon\epsilon_0\epsilon_1 ... \epsilon_p P^2 \frac{q}{2(d-2)} e^{2(1-p)U}\tau^{-2s} \end{gathered} \tag{4.1.1}$$

and

$$\begin{gathered} R_{\tau\tau} = \frac{p-1}{q}\left(\ddot{U} + \frac{\dot{U}}{\tau}\right) \\ \Rightarrow \ddot{U} - \left[1 - 2\left(\frac{d-2}{q}\right)l\right]\frac{\dot{U}}{\tau} - \frac{(p-1)(d-2)}{q}\dot{U}^2 = \frac{p-1}{q}\left(\ddot{U} + \frac{\dot{U}}{\tau}\right) \\ \Rightarrow \ddot{U} + (1-2l)\frac{\dot{U}}{\tau} + (p-1)\dot{U}^2 = 0 \end{gathered} \tag{4.1.2}$$

This is a nonlinear differential equation. To solve this, we take

$$\begin{gathered} W = e^{(p-1)U} \\ \Rightarrow (p-1)\dot{U} = \frac{\dot{W}}{W} \\ \Rightarrow \ddot{U} + (p-1)\dot{U}^2 = \frac{\ddot{W}}{W(p-1)} \end{gathered} \tag{4.1.3}$$

Then the differential equation becomes

$$
\begin{aligned}
&\ddot{W} - \frac{1-2l}{\tau}\dot{W} = 0 \\
&\Rightarrow \dot{W} = a_1 \tau^{1-2l} \\
&\Rightarrow W = \frac{a_1}{2l}\tau^{2l} + a_2 = c_1\tau^{2l} + c_2 \\
&\Rightarrow e^U = \left(c_1 + c_2\tau^{2l}\right)^{\frac{1}{p-1}}
\end{aligned}
\tag{4.1.4}
$$

Therefore

$$
U = \frac{1}{p-1}\ln\left(c_1 + c_2\tau^{2l}\right) \tag{4.1.5}
$$

Then

$$
\begin{aligned}
\dot{U} &= \frac{1}{p-1}\frac{1}{c_1 + c_2\tau^{2l}} 2c_2 l\tau^{2l-1} \\
\ddot{U} &= \frac{1}{p-1}\left[\frac{2c_2 l(2l-1)\tau^{2l-2}}{c_2\tau^{2l} + c_1} - \frac{4c_2^2 l^2\tau^{4l-2}}{\left(c_1 + c_2\tau^{2l}\right)^2}\right]
\end{aligned}
\tag{4.1.6}
$$

Then we can write

$$
\ddot{U} + \frac{\dot{U}}{\tau} = \frac{4l^2 c_1 c_2 \tau^{2l-2}}{(p-1)W^2} \tag{4.1.7}
$$

Now using Equation (4.1.1)

$$
\begin{aligned}
\ddot{U} + \frac{\dot{U}}{\tau} &= -\epsilon\epsilon_0\epsilon_1 ... \epsilon_p P^2 \frac{q}{2(d-2)} e^{2(1-p)U}\tau^{-2s} \\
8\frac{(d-2)l^2 c_1 c_2}{(p-1)q} &+ \epsilon\epsilon_0\epsilon_1 ... \epsilon_p P^2 = 0.
\end{aligned}
\tag{4.1.8}
$$

Thus we found solutions for $U(\tau), V(\tau)$ and P.

4.2 Gravity theory with m-form and a coupled dilaton scalar

The action in Equation (3.5.1) gives us the equations of motion

$$
\begin{aligned}
R_{\mu\nu} &= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{\epsilon e^{\beta\phi}}{2(p-1)!}\left[F_{\mu\alpha_2 ... \alpha_p}F_\nu{}^{\alpha_2 ... \alpha_p} - g_{\mu\nu}F^2\frac{p-1}{p(d-2)}\right] \\
\partial_\mu\left(\sqrt{|g|}e^{\beta\phi}F^{\mu\nu_2 ... \nu_p}\right) &= 0 \\
\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\partial^\mu\phi\right) &= -\frac{\beta}{2p!}\epsilon e^{\beta\phi}F_{\mu_1 ... \mu_p}F^{\mu_1 ... \mu_p}
\end{aligned}
\tag{4.2.1}
$$

Now we can put the flux and metric ansatz into them and get

$$
\begin{aligned}
R_{x_i x_i} &= \frac{\epsilon e^{\beta\phi}}{2(p-1)!}\left[(p-1)!E\epsilon_i P^2 e^{-2(p-1)U}\tau^{-2(s-a_i)} - \frac{p-1}{p(d-2)}p! g_{x_i x_i}P^2 E e^{-2pU}\tau^{-2s}\right] \\
\Rightarrow -\epsilon_0\epsilon_i\tau^{2a_i}\left(\ddot{U} + \frac{\dot{U}}{\tau}\right) &= \frac{q}{2(d-2)}\epsilon\epsilon_i\epsilon_1 ... \epsilon_p P^2 e^{-2U(p-1)}\tau^{-2(s-a_i)}e^{\beta\phi} \\
\Rightarrow \ddot{U} + \frac{\dot{U}}{\tau} &= -\frac{q}{2(d-2)}j
\end{aligned}
\tag{4.2.2}
$$

here

$$
j = \epsilon_0\epsilon\epsilon_1 ... \epsilon_p P^2 e^{-2U(p-1)+\beta\phi}\tau^{-2s}. \tag{4.2.3}
$$

And

$$
\begin{aligned}
R_{\tau\tau} &= \frac{1}{2}\dot{\phi}^2 - \epsilon\epsilon_i\epsilon_0 E P^2 e^{\beta\phi}\frac{p-1}{2(d-2)}e^{-2U(p-1)}\tau^{-2s} \\
&= \frac{1}{2}\dot{\phi}^2 - \frac{p-1}{2(d-2)}j
\end{aligned}
\tag{4.2.4}
$$

Now, we write the third equation using $\sqrt{|g|} = Ce^{2U}\tau$

$$\begin{aligned}\frac{1}{Ce^{2U}\tau}\partial_\tau\left(Ce^{2U}\tau\epsilon_0 e^{-2U}\partial_\tau\phi\right) &= \frac{\beta}{2p!}\epsilon e^{\beta\phi}p!P^2\left(\epsilon_1...\epsilon_p\right)e^{-2pU}\tau^{-2s}\\ \Rightarrow \frac{e^{-2U}}{\tau}\epsilon_0\left(\tau\ddot{\phi}+\dot{\phi}\right) &= \frac{\beta}{2}\epsilon\epsilon_1...\epsilon_p e^{-2pU+\beta\phi}\tau^{-2s}\\ \Rightarrow \ddot{\phi}+\frac{\dot{\phi}}{\tau} &= \frac{\beta}{2}\epsilon\epsilon_0\epsilon_1...\epsilon_p e^{-2U(p-1)+\beta\phi}\tau^{-2s}\\ \Rightarrow \ddot{\phi}+\frac{\dot{\phi}}{\tau} &= \frac{\beta}{2}j\end{aligned} \tag{4.2.5}$$

Now, using Equation (4.2.5) and Equation (4.2.2)

$$\begin{aligned}\ddot{U}+\frac{\dot{U}}{\tau} &= -\frac{q}{\beta(d-2)}\left(\ddot{\phi}+\frac{\dot{\phi}}{\tau}\right)\\ \Rightarrow \phi &= -\frac{\beta(d-2)}{q}U\end{aligned} \tag{4.2.6}$$

We can write

$$e^{-2U(p-1)+\beta\phi} = e^{-2\mu U} \tag{4.2.7}$$

with $\mu = p-1+(d-2)\frac{\beta^2}{2q}$.

Then Equation (4.2.2) becomes

$$\ddot{U}+\frac{\dot{U}}{\tau} = -\epsilon\epsilon_0\epsilon_1...\epsilon_p\frac{q}{2(d-2)}P^2e^{-2\mu U}\tau^{-2s} \tag{4.2.8}$$

Now, we write Equation (4.2.4)

$$\begin{aligned}\ddot{U}-\left[1-2\left(\frac{d-2}{q}\right)l\right]\frac{\dot{U}}{\tau}-\frac{(p-1)(d-2)}{q}\dot{U}^2 &= \frac{1}{2}\left(\frac{\beta(d-2)}{q}\right)^2\dot{U}^2-\frac{p-1}{q}\left(\ddot{U}+\frac{\dot{U}}{\tau}\right)\\ \Rightarrow \ddot{U}+\frac{\dot{U}}{\tau}(1-2l)+\dot{U}^2\mu &= 0\end{aligned} \tag{4.2.9}$$

This non linear differential equation has the same form that we solved earlier in Equation (4.1.2). Hence the solution is

$$e^U = \left(c_1+c_2\tau^{2l}\right)^{\frac{1}{\mu}} \tag{4.2.10}$$

We can write

$$\ddot{U}+\frac{\dot{U}}{\tau} = \frac{4l^2c_1c_2\tau^{2l-2}}{\mu e^{2\mu U}} \tag{4.2.11}$$

Now putting this back to Equation (4.2.8), we get

$$\begin{aligned}\frac{4l^2c_1c_2\tau^{2l-2}}{\mu e^{2\mu U}}+\epsilon\epsilon_0\epsilon_1...\epsilon_p\frac{q}{2(d-2)}P^2e^{-2\mu U}\tau^{-2s} &= 0\\ P^2+\epsilon\epsilon_0\epsilon_1...\epsilon_p c_1c_2\frac{8(d-2)l^2}{\mu} &= 0\end{aligned} \tag{4.2.12}$$

Hence, we found expressions for ϕ, U and P.

5 Towards more general solutions

We consider the metric ansatz

$$\mathrm{d}s^2 = e^{\alpha(t)}\left(-\mathrm{d}t^2+e^{\gamma_1(t)}\,\mathrm{d}{x_1}^2+e^{\gamma_2(t)}\,\mathrm{d}{x_2}^2+e^{\gamma_3(t)}\,\mathrm{d}{x_3}^2\right) \tag{5.1}$$

with scalar $\phi(t)$ and a two form flux

$$\begin{aligned}F_2 = f_1(x)q_1(t)\,\mathrm{d}t\wedge\mathrm{d}x_1+f_2(x)q_2(t)\,\mathrm{d}t\wedge\mathrm{d}x_2+f_3(x)q_3(t)\,\mathrm{d}t\wedge\mathrm{d}x_3\\ +g_1(x)h_1(t)\,\mathrm{d}x_1\wedge\mathrm{d}x_2+g_2(x)h_2(t)\,\mathrm{d}x_2\wedge\mathrm{d}x_3+g_3(x)h_3(t)\,\mathrm{d}x_3\wedge\mathrm{d}x_1\end{aligned} \tag{5.2}$$

Here, the metric

$$g_{\mu\nu} = \mathrm{diag}(-e^{\alpha}, e^{\alpha+\gamma_1}, e^{\alpha+\gamma_2}, e^{\alpha+\gamma_3}) \tag{5.3}$$

and

$$\sqrt{|g|} = e^{2\alpha+\frac{1}{2}(\gamma_1+\gamma_2+\gamma_3)} \tag{5.4}$$

The non-zero Ricci tensor

$$R_{tt} = -\frac{3}{2}\ddot{\alpha} - \frac{1}{2}\sum_i \ddot{\gamma}_i - \frac{1}{4}\sum_i \dot{\gamma}_i^2 - \frac{1}{4}\dot{\alpha}\sum_i \dot{\gamma}_i \tag{5.5}$$

$$R_{x_ix_i} = \frac{e^{\gamma_i}}{2}\left[\ddot{\alpha} + \ddot{\gamma}_i + \dot{\alpha}^2 + \dot{\alpha}\left(\frac{1}{2}\sum_j \dot{\gamma}_j + \dot{\gamma}_i\right) + \frac{1}{2}\dot{\gamma}_i\sum_j \dot{\gamma}_j\right] \tag{5.6}$$

and the mixed components give zero.

Two form flux

Now, we consider the flux ansatz. The non zero components are $F_{0i} = f_i q_i, F_{12} = g_1 h_1, F_{23} = g_2 h_2,$ and $F_{31} = g_3 h_3$. Raising the indices using the metric gives

$$F^{0i} = -e^{-2\alpha-\gamma_i} f_i q_i \quad \& \quad F^{12} = e^{-2\alpha-\gamma_1-\gamma_2} g_1 h_1 \tag{5.7}$$

Then, we can compute

$$F^2 = 2\left[-\sum_i e^{-2\alpha-\gamma_i}(f_iq_i)^2 + e^{-2\alpha-\gamma_1-\gamma_2}(g_3h_3)^2 + e^{-2\alpha-\gamma_2-\gamma_3}(g_1h_1)^2 + e^{-2\alpha-\gamma_3-\gamma_1}(g_2h_2)^2\right] \tag{5.8}$$

$$F_{0\rho}F_0^{\rho} = \sum_i e^{-\alpha-\gamma_i}(f_iq_i)^2 \tag{5.9}$$

and

$$F_{1\rho}F_1^{\rho} = -e^{-\alpha-\gamma_1}(f_1q_1)^2 + e^{-\alpha-\gamma_2}(g_3h_3)^2 + e^{-\alpha-\gamma_3}(g_2h_2)^2 \tag{5.10}$$

$$F_{2\rho}F_2^{\rho} = -e^{-\alpha-\gamma_2}(f_2q_2)^2 + e^{-\alpha-\gamma_3}(g_1h_1)^2 + e^{-\alpha-\gamma_1}(g_3h_3)^2 \tag{5.11}$$

$$F_{3\rho}F_3^{\rho} = -e^{-\alpha-\gamma_3}(f_3q_3)^2 + e^{-\alpha-\gamma_1}(g_2h_2)^2 + e^{-\alpha-\gamma_2}(g_1h_1)^2 \tag{5.12}$$

Variation of action

Our action is

$$S = \int \mathrm{d}^4x\, \sqrt{|g|}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\varepsilon}{4}e^{\beta\phi}F^2\right) \tag{5.13}$$

Varying the action, we get the equation of motions

$$R_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{\varepsilon e^{\beta\phi}}{2}\left(F_{\mu\rho}F_\nu^{\rho} - \frac{1}{4}g_{\mu\nu}F^2\right) \tag{5.14}$$

$$\partial_\mu\left(\sqrt{|g|}e^{\beta\phi}F^{\mu\nu}\right) = 0 \tag{5.15}$$

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\partial^\mu\phi\right) = \frac{\beta\varepsilon}{4}e^{\beta\phi}F^2 \tag{5.16}$$

Flux coefficients

Equation (5.15) gives us the constraints on $f(x), q(t), g(x),$ and $h(t)$.

- Taking $\nu = i$

$$\begin{aligned}\partial_t\left(e^{\beta\phi+\frac{1}{2}\sum_i\gamma_i-\gamma_i}f_iq_i\right) + \partial_{x_1}\left(e^{\beta\phi+\frac{1}{2}\sum_i\gamma_i-\gamma_2-\gamma_1}s_{12}\right) +& \\ \partial_{x_2}\left(e^{\beta\phi+\frac{1}{2}\sum_i\gamma_i-\gamma_3-\gamma_2}s_{23}\right) + \partial_{x_3}\left(e^{\beta\phi+\frac{1}{2}\sum_i\gamma_i-\gamma_1-\gamma_3}s_{31}\right) &= 0\end{aligned} \tag{5.17}$$

here $s_{12} = g_1h_1, s_{23} = g_2h_2, s_{31} = g_3h_3$

- Taking $\nu = 0$

$$\sum_i \partial_{x_i}\left(e^{\beta\phi+\frac{1}{2}\sum_i\gamma_i-\gamma_i}f_iq_i\right) = 0 \tag{5.18}$$

This requires $f(x)$ to be a constant. It follows from Equation (5.17)

$$\partial_t\left(e^{\beta\phi+\frac{1}{2}\sum_i\gamma_i-\gamma_i}f_iq_i\right)=0 \quad \& \quad \partial_{x_j}s_{ij}=0 \quad (i\neq j) \tag{5.19}$$

This implies, g_k, h_k to be constants and

$$q_i(t)=Q_ie^{-\beta\phi-\frac{1}{2}\sum_i\gamma_i+\gamma_i} \tag{5.20}$$

Equation of motions

Now, we introduce a set of new variables and assumptions to simplify our system of equation. First, we fix the time gauge $\alpha(t)=0$ and consider $X=\frac{1}{2}\sum_i\gamma_i$. Then the metric becomes $\sqrt{|g|}=e^X$.

Now the Equation (5.16) becomes

$$\ddot{\phi}+\dot{X}\dot{\phi}=\frac{\varepsilon\beta}{2}e^{\beta\phi}\left[-\sum_i e^{-\gamma_i}q_i^2+e^{-\gamma_1-\gamma_2}+e^{-\gamma_2-\gamma_3}+e^{-\gamma_3-\gamma_1}\right] \tag{5.21}$$

Equation (5.14) becomes

$$\begin{aligned}R_{tt}&=\frac{1}{2}\dot{\phi}^2+\frac{\varepsilon}{2}e^{\beta\phi}\left[\frac{1}{2}\sum_i e^{-\gamma_i}q_i^2+\frac{1}{2}(e^{-\gamma_1-\gamma_2}+e^{-\gamma_2-\gamma_3}+e^{-\gamma_3-\gamma_1})\right]\\ -\ddot{X}-\frac{1}{4}\sum_i\dot{\gamma}_i^2&=\frac{1}{2}\dot{\phi}^2+\frac{\varepsilon}{2}e^{\beta\phi}\left[\frac{1}{2}\sum_i e^{-\gamma_i}q_i^2+\frac{1}{2}(e^{-\gamma_1-\gamma_2}+e^{-\gamma_2-\gamma_3}+e^{-\gamma_3-\gamma_1})\right]\end{aligned} \tag{5.22}$$

and

$$\begin{aligned}R_{ii}&=\frac{\varepsilon e^{\beta\phi}}{2}\left(F_{i\rho}F_i^{\rho}-\frac{1}{4}g_{ii}F^2\right)\\ e^{\gamma_i}\left[\frac{1}{2}\ddot{\gamma}_i+\frac{1}{2}\dot{\gamma}_i\dot{X}\right]&=\frac{\varepsilon e^{\beta\phi}}{2}\left[-e^{-\gamma_i}q_i^2+e^{-\gamma_j}+e^{-\gamma_k}\right.\\ &\left.\quad+\frac{1}{2}\left(-\sum_m e^{\gamma_i-\gamma_m}q_m^2+e^{\gamma_i-\gamma_1-\gamma_2}+e^{\gamma_i-\gamma_2-\gamma_3}+e^{\gamma_i-\gamma_3-\gamma_1}\right)\right]\end{aligned} \tag{5.23}$$

Comments

The EOMs have four unknowns $\phi,\gamma_1,\gamma_2,\gamma_3$. We have four second-order differential equations here. Thus, without further constraints, it is not always possible to find closed-form solutions for the unknowns. Also, since we have a coupled system, it is not possible to separate the variables. To solve the system, we might need to impose further conditions. We can consider the isotropic case $\gamma_1=\gamma_2=\gamma_3$ and also all the Q_i to be equal. Moreover, if we take q_i to be zero, we should be able to recover the Kasner-like solutions. But in this current format and considering anisotropy, we are uncertain about the solvability of the EOMs.

6 Discussion

In this report, we have explicitly derived time dependent and static solutions for generalised Kasner metric anstaz including matter fields–form flux and coupled dilaton. We found closed form solutions for dilaton scalar, the form, and the metric coefficients. Then we attempted an even general metric in hopes of finding non-power law solutions. However, we were not able to derive closed form solution for this case.

The action of primary importance in our work is Equation (3.5.1). This action is a model for bosonic fields of many supergravity theories. We found non-trivial solution for it. This method can be extended to derive solutions for theories of type IIA, type IIA*, type IIB, type IIB*, and type IIB' supergravity theories. In [4], Hull has presented a list of these theories with corresponding time signatures and signs for kinetic gauge terms.

Significance of working with Kasner dynamics is it provides an unifying language to deal with many different problems since they give an exact description of anisotropic expansion which work across dimensions. In [5], Apers, and Conlon, and Mosny discuss kination in 4D early universe cosmology. The expansion due to kinetic energy of rolling moduli is defined as kination. It has an interpretation in 10D as a Kasner background. The 4D scale factor $\alpha \propto t^{\frac{1}{3}}$ and the evolving moduli uplifts to a higher dimensional Kasner metric with rolling dilaton. Another interesting result in [5] is how 10D perturbations around the Kasner background map to 4D matter perturbations in the kination phase, indicating that information can pass back and forth between the 10D and 4D pictures in a controlled way. Sugiyama, Yamamoto, and Kobayashi show in [6] that Kasner branes can be used to study quantum fields and gravitational waves in anisotropic universes. Moreover, Andriot, Cribiori and Riet find that in higher dimensional string theory, rolling solutions supported by fluxes and module naturally give rise to Kasner-like time scaling. [7] The authors reviews the scale separation and show that static solutions face strong no-go theorems. However, the Kasner branes provide a way around to the no-go barriers.

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A Review

A.1 Lagrangian Formalism

If we define positions of N particles with coordinates x^A where $A = 1, ..., 3N$. Here, the number of degrees of freedom is $3N$, which forms a *configuration space* C. Each point in C represents the position of all N particles, and the time evolution of a particle is a curve in C.

We write the Lagrangian as

$$L(x^A, \dot{x}^A) = T(\dot{x}^A) - V(x^A) \qquad \text{A.1.1}$$

Here $T(\dot{x}^A) = \frac{1}{2}\sum_A m_A \dot{x}^A$ is the kinetic energy and V is the potential energy.

Now we consider all smooth paths $x^A(t)$ in C with fixed endpoints so that

$$x^A(t_i) = x^A_{initial} \quad and \quad x^A(t_f) = x^A_{final} \qquad \text{A.1.2}$$

We can define a functional called action

$$S[x^A(t)] = \int_{t_i}^{t_f} L(x^A, \dot{x}^A)\, \mathrm{d}t \qquad \text{A.1.3}$$

Principle of Least Action Theorem 1.1.1

The actual path taken by the system is an extremum of S.

Proof: If we vary the given path slightly

$$x^A(t) \to x^A(t) + \delta x^A(t) \qquad \text{A.1.4}$$

and we fix the end points by $\delta x^A(t_i) = \delta x^A(t_f) = 0$. Then

$$\begin{aligned}
\delta S &= \int_{t_i}^{t_f} \delta L \,\mathrm{d}t \\
&= \int_{t_i}^{t_f} \left[\frac{\partial L}{\partial x^A}\delta x^A + \frac{\partial L}{\partial \dot{x}^A}\delta \dot{x}^A\right] \mathrm{d}t \\
&= \int_{t_i}^{t_f} \left[\frac{\partial L}{\partial x^A}\delta x^A + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^A}\delta x^A\right) - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^A}\right)\delta x^A\right] \mathrm{d}t \\
&= \int_{t_i}^{t_f} \mathrm{d}t \left[\frac{\partial L}{\partial x^A} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^A}\right)\right]\delta x^A + \left[\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^A}\right)\delta x^A\right]_{t_i}^{t_f}
\end{aligned} \tag{A.1.5}$$

Since $\delta S = 0$

$$\therefore \frac{\partial L}{\partial x^A} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}^A}\right) = 0 \tag{A.1.6}$$

These are the Euler-Lagrange equations.

We can get Newton's equation from this. We know

$$\dot{p}_A = -\frac{\partial V}{\partial x^A} \tag{A.1.7}$$

where $p_A = m_A \dot{x}^A$.

Then, Newton's equation follows from the E-L equations

$$\begin{aligned}
-\frac{\partial V}{\partial x^A} - \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}^A}\right) = 0 \\
\Rightarrow -\frac{\partial V}{\partial x^A} = \frac{\partial}{\partial t}(m_A \dot{x}^A) = \dot{p}_A
\end{aligned} \tag{A.1.8}$$

A.2 Classical Field Theory

We shift from generalised coordinates $q_a(t)$ to defining fields at every point of space and time. The dynamics of fields are described by

$$\varphi_a(\vec{x}, t) \tag{A.2.1}$$

where a and $\vec{x}$ are both labels. This gives rise to an infinite number of degrees of freedom in field theory.

We can write the Lagrangian

$$L(t) = \int \mathrm{d}^3x \ \mathcal{L}(\varphi_a, \partial_\mu \varphi_a) \tag{A.2.2}$$

$\mathcal{L}$ is called the Lagrangian density. We write the action

$$S = \int \mathrm{d}^4x \mathcal{L} \tag{A.2.3}$$

If we vary the action

$$\begin{aligned}
\delta S &= \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi_a}\delta\varphi_a + \frac{\partial \mathcal{L}}{\partial(\partial_u \varphi_a)}\delta(\partial_\mu \varphi_a)\right] \\
&= \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi_a}\delta\varphi_a + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}\delta\varphi_a\right) - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}\right)\delta\varphi_a\right] \\
&= \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}\right)\right]\delta\varphi_a + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}\delta\varphi_a\right)
\end{aligned} \tag{A.2.4}$$

With the boundary conditions $\delta\varphi_a(\vec{x}, t_i) = \delta\varphi_a(\vec{x}, t_f) = 0$ the last term vanishes. And to satisfy the *least action principle* $\delta S = 0$, we get the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \varphi_a} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)}\right) = 0 \tag{A.2.5}$$

Klein-Gordon Equation

The Lagrangian for a real scalar field $\varphi(\vec{x}, t)$

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2 \tag{A.2.6}$$

here, $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Now we have

$$\frac{\partial\mathcal{L}}{\partial\varphi} = -m^2\varphi \quad \& \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \partial^\mu\varphi = (\dot{\varphi}, -\nabla\varphi) \tag{A.2.7}$$

Then the E-L equation

$$\begin{aligned} \partial_\mu\partial^\mu\varphi + m^2\varphi = 0 \\ \ddot{\varphi} - \nabla^2\varphi + m^2\varphi = 0 \end{aligned} \tag{A.2.8}$$

This is the Klein-Gordon Equation.

Maxwell's Equation

Maxwell's Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2 \tag{A.2.9}$$

We define the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $A^\mu(\vec{x}, t) = \left(\varphi, \vec{A}\right)$ Then we can write

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= \partial_\mu A_\nu\partial^\mu A^\nu - \partial_\mu A_\nu\partial^\nu A^\mu - \partial_\nu A_\mu\partial^\mu A^\nu + \partial_\nu A_\mu\partial^\nu A^\mu \\ &= 2\partial_\mu A_\nu\partial^\mu A^\nu - 2\partial_\mu A_\nu\partial^\nu A^\mu \\ &= 2\partial_\mu A_\nu\partial^\mu A^\nu - 2\eta_{\mu\nu}\partial_\mu A^\mu\eta^{\alpha\nu}\partial_\alpha A^\mu \\ &= 2\partial_\mu A_\nu\partial^\mu A^\nu - 2\partial_\mu A^\mu\partial_\mu A^\mu \end{aligned} \tag{A.2.10}$$

Hence,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{A.2.11}$$

Now, we can derive Maxwell's equations using the field strength tensor. The electric and magnetic field

$$E_i = F_{0i} \quad \& \quad B_i = \frac{1}{2}\varepsilon_{ijk}F^{jk} \tag{A.2.12}$$

Then

$$\vec{E} = -\boldsymbol{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t} \tag{A.2.13}$$

and

$$\begin{aligned} B_i &= \frac{1}{2}\varepsilon_{ijk}\left(\partial^j A^k - \partial^k A^j\right) \\ &= \frac{1}{2}\varepsilon_{ijk}\partial^j A^k - \frac{1}{2}\varepsilon_{ijk}\partial^k A^j \\ &= \frac{1}{2}\varepsilon_{ijk}\partial^j A^k + \frac{1}{2}\varepsilon_{ijk}\partial^j A^k \\ &= (\boldsymbol{\nabla} \times A)_i \end{aligned} \tag{A.2.14}$$

Then, using vector identities $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times A) = 0$ and $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} A) = 0$, we get, Gauss and Faraday's laws

$$\boldsymbol{\nabla} \cdot B = 0 \tag{A.2.15}$$

and

$$\begin{aligned} \frac{\partial B}{\partial t} &= \boldsymbol{\nabla} \times \frac{\partial\vec{A}}{\partial t} \\ &= \boldsymbol{\nabla} \times (-E - \boldsymbol{\nabla}\varphi) \\ &= -\boldsymbol{\nabla} \times E \end{aligned} \tag{A.2.16}$$

Now the E-L equation

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = -\partial_\mu\partial^\mu A^\nu + \partial^\nu\partial_\mu A^\mu = -\partial_\mu F^{\mu\nu} = 0 \tag{A.2.17}$$

This follows

$$\boldsymbol{\nabla}\cdot E = 0 \tag{A.2.18}$$

and

$$\begin{aligned}(\boldsymbol{\nabla}\times B)_i &= \varepsilon_{ijk}\partial_j B_k \\ &= \partial_j\varepsilon_{ijk}\frac{1}{2}\varepsilon_{kab}F^{ab} \\ &= \partial_j\frac{1}{2}\varepsilon_{ijk}\varepsilon_{abk}F^{ab} \\ &= \partial_j\frac{1}{2}(\delta_{ia}\delta_{jb}-\delta_{ib}\delta_{ja})F^{ab} \\ &= \partial_j\frac{1}{2}(F^{ij}-F^{ji}) \\ &= -\partial_j F^{ji} \\ &= \partial_j F_{ji}\end{aligned} \tag{A.2.19}$$

Setting $j = 0$,

$$\boldsymbol{\nabla}\times B = \frac{\partial E}{\partial t} \tag{A.2.20}$$

Hence, we derived all four Maxwell's equations in vacuum.

A.3 Einstein-Hilbert Action

We can write the action

$$S = \int \mathrm{d}^4x\sqrt{-g}R \tag{A.3.1}$$

here $g \equiv \det(g_{\mu\nu})$ and Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$.

If we vary the action

$$\delta S = \int \mathrm{d}^4x\left[\delta\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\,\delta g^{\mu\nu}\,R_{\mu\nu} + \sqrt{-g}g^{\mu\nu}\,\delta R_{\mu\nu}\right] \tag{A.3.2}$$

here the last term is a total derivative $g^{\mu\nu}\,\delta R_{\mu\nu} = \nabla_\mu X^\mu$, hence we can ignore that term.

Now we know that any diagonalizable matrix M satisfies the identity

$$\begin{aligned}\ln\det(M) &= \mathrm{Tr}(\ln M) \\ \Rightarrow \frac{1}{\det M}\,\delta\det M &= \mathrm{Tr}\left(\frac{1}{M}\,\delta M\right)\end{aligned} \tag{A.3.3}$$

Taking $M = g_{\mu\nu}$ and $\det g_{\mu\nu} = g$

$$g^{-1}\,\delta g = g^{\mu\nu}\,\delta g_{\mu\nu} \tag{A.3.4}$$

Hence

$$\begin{aligned}\delta\sqrt{-g} &= \frac{1}{2\sqrt{-g}}\,\delta g \\ &= \frac{1}{2\sqrt{-g}}gg^{\mu\nu}\,\delta g_{\mu\nu} \\ &= -\frac{\sqrt{-g}}{2}g_{\mu\nu}\,\delta g^{\mu\nu}\end{aligned} \tag{A.3.5}$$

Hence, the variation of action

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu}\right)\delta g^{\mu\nu} \tag{A.3.6}$$

Following the least action principle, we get

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{A.3.7}$$

This is the Einstein equation in vacuum where $G_{\mu\nu}$ is the Einstein tensor.

B Differential Forms

We attempt to provide a brief overview of differential forms based on Eguchi, Gilkey, and Hanson in [8].

Differential Forms Definition 2.1

Differential forms are the totally antisymmetric covariant tensor fields.

Wedge Product Definition 2.2

We define the wedge product or exterior product as the antisymmetric tensor product of basis elements of cotangent space.

$$\mathrm{d}x \wedge \mathrm{d}y = \frac{1}{2}(\mathrm{d}x \otimes \mathrm{d}y - \mathrm{d}y \otimes \mathrm{d}x) = -\,\mathrm{d}y \wedge \mathrm{d}x \tag{B.1}$$

By definition

$$\mathrm{d}x \wedge \mathrm{d}x = 0 \tag{B.2}$$

The wedge product has the following properties.

- Linearity

$$\alpha_p \wedge \left(a\beta_q + b\gamma_r\right) = a\left(\alpha_p \wedge \beta_q\right) + b\left(\alpha_p \wedge \gamma_r\right) \tag{B.3}$$

- Associativity

$$\alpha_p \wedge \left(\beta_q \wedge \gamma_r\right) = \left(\alpha_p \wedge \beta_q\right) \wedge \gamma_r \tag{B.4}$$

- Graded Commutativity

$$\alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p \tag{B.5}$$

p-forms Definition 2.3

We define a p-from as

$$\alpha = \frac{1}{p!} \alpha_{\mu_1 \dots \mu_p} \mathrm{d}^{\mu_1} \wedge \dots \wedge \mathrm{d}^{\mu_p} \tag{B.6}$$

Exterior Derivative Definition 2.4

The exterior derivative is a linear map such that it takes a p form to (p+1) form

$$\mathrm{d} : \Lambda^p \to \Lambda^{p+1} \tag{B.7}$$

We define

$$\mathrm{d}\alpha = \frac{1}{p!} \partial_{[\nu} \alpha_{\mu_1 \dots \mu_p]} \, \mathrm{d}x^\nu \wedge \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_p} \tag{B.8}$$

We can write the components of $\mathrm{d}\alpha$ as

$$\mathrm{d}\alpha = \frac{1}{(p+1)!} (\mathrm{d}\alpha)_{\mu_1 \dots \mu_{p+1}} \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_{p+1}} \tag{B.9}$$

We have

$$(\mathrm{d}\alpha)_{\mu_1 \dots \mu_{p+1}} = (p+1) \partial_{[\mu_1} \alpha_{\mu_2 \dots \mu_{p+1}]} \tag{B.10}$$

It has the following properties

- Leibniz rule

$$\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f \tag{B.11}$$

- $\mathbb{R}$ linearity

$$\mathrm{d}(af + bg) = a\,\mathrm{d}f + b\,\mathrm{d}g \tag{B.12}$$

- Nilpotency

$$\mathrm{d}\,\mathrm{d}\alpha = 0 \tag{B.13}$$

Proof.

$$\begin{aligned}\mathrm{d}\,\mathrm{d}\alpha &= \frac{1}{(p+2)!}(\mathrm{d}\,\mathrm{d}\alpha)_{\mu_1\cdots\mu_{p+2}}\,\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{p+2}} \\ &= \frac{1}{(p+2)!}(p+2)(p+1)\partial_{[\mu_1}\partial_{\mu_2}\alpha_{\mu_3\cdots\mu_{p+2}]}\,\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{p+2}} \\ &= \frac{1}{p!}\partial_{[\mu_1}\partial_{\mu_2]}\alpha_{\mu_3\cdots\mu_{p+2}}\,\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_{p+2}}\end{aligned} \tag{B.14}$$

Now

$$\partial_{[\mu_1}\partial_{\mu_2]} = \frac{1}{2}\left(\partial_{\mu_1}\partial_{\mu_2} - \partial_{\mu_2}\partial_{\mu_1}\right) = 0 \tag{B.15}$$

Hence,

$$\mathrm{d}\,\mathrm{d}\alpha = 0 \tag{B.16}$$

- Exterior derivative of wedge products of a p-form and a q-form is

$$\mathrm{d}\left(\alpha_p\wedge\beta_q\right) = \mathrm{d}\alpha_p\wedge\beta_q + (-1)^p\alpha_p\wedge\mathrm{d}\beta_q \tag{B.17}$$

The dimension of the space of p-forms and (n-p)-form is the same. This tells us there exists a duality between them.

Hodge Star Definition 2.5

We define the hodge star or the duality transformation as

$$\star : \Lambda^p \to \Lambda^{n-p} \tag{B.18}$$

In a flat euclidean space

$$\star\left(\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p}\right) = \frac{1}{(n-p)!}\varepsilon_{i_1 i_2\ldots i_p i_{p+1}\cdots i_n}\,\mathrm{d}x^{i_{p+1}}\wedge\ldots\wedge\mathrm{d}x^{i_n} \tag{B.19}$$

Repeating the $\star$ operator on a p-form gives

$$\star\star\,\alpha = (-1)^{p(n-p)}\alpha \tag{B.20}$$

Proof.

$$\begin{aligned}\star\,\alpha &= \frac{1}{p!(n-p)!}\alpha_{\mu_1\cdots\mu_p}\varepsilon_{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_n}\,\mathrm{d}x^{\mu_{p+1}}\wedge\ldots\wedge\mathrm{d}x^{\mu_n} \\ \star\star\,\alpha &= \frac{1}{p!(n-p)!}\alpha_{\mu_1\cdots\mu_p}\varepsilon_{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_n}\frac{1}{p!}\varepsilon_{\mu_{p+1}\cdots\mu_n\mu_1\cdots\mu_p}\,\mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_p} \\ &= \frac{1}{p!(n-p)!}\varepsilon_{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_n}\varepsilon_{\mu_{p+1}\cdots\mu_n\mu_1\cdots\mu_p}\alpha \\ &= \frac{1}{p!(n-p)!}\varepsilon_{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_n}(-1)^{p(n-p)}\varepsilon_{\mu_1\cdots\mu_p\mu_{p+1}\cdots\mu_n}\alpha \\ &= (-1)^{p(n-p)}\frac{1}{p!(n-p)!}(n-p)!\,\delta^{\mu_1\cdots\mu_p}_{\mu_1\cdots\mu_p}\alpha \\ &= (-1)^{p(n-p)}\frac{1}{p!}p!\alpha \\ &= (-1)^{p(n-p)}\alpha\end{aligned} \tag{B.21}$$

□

Inner Product Definition 2.6

We define the inner product as integral

$$(\alpha_p, \beta_p) = \int_M \alpha_p \wedge \star\, \beta_p \tag{B.22}$$

It has following property

$$(\alpha_p, \beta_p) = (\beta_p, \alpha_p) \tag{B.23}$$

where

$$\alpha_p \wedge \star\, \beta_p = \beta_p \wedge \star\, \alpha_p \tag{B.24}$$

Adjoint of Exterior Derivative Definition 2.7

We define the adjoint of d as

$$\delta = (-1)^{np+n+1} \star \mathrm{d} \star \tag{B.25}$$

The adjoint derivative reduces the degree of the differential form by one unit.

$$\delta : \Lambda^p \to \Lambda^{p-1} \tag{B.26}$$

Then we can show

$$(\alpha_p, \mathrm{d}\beta_{p-1}) = (\delta\alpha_p, \beta_{p-1}) \tag{B.27}$$

Proof. We know

$$\begin{aligned} \mathrm{d}(\alpha_p \wedge \beta_q) &= \mathrm{d}\alpha_p \wedge \beta_p + (-1)^p \alpha_p \wedge \mathrm{d}\beta_q \\ \mathrm{d}(\beta_{p-1} \wedge \star\, \alpha_p) &= \mathrm{d}\beta_{p-1} \wedge \star\, \alpha_p + (-1)^{p-1} \beta_{p-1} \wedge \mathrm{d} \star \alpha_p \end{aligned} \tag{B.28}$$

Now, if we integrate

$$\begin{aligned} \int_M \mathrm{d}(\beta_{p-1} \wedge \star\, \alpha_p) &= 0 \\ \int_M \mathrm{d}\beta_{p-1} \wedge \star\, \alpha_p &= -\int_M (-1)^{p-1} \beta_{p-1} \wedge \mathrm{d} \star \alpha_p \\ (\mathrm{d}\beta_{p-1}, \alpha_p) &= \int_M (-1)^p \beta_{p-1} \wedge \mathrm{d} \star \alpha_p \end{aligned} \tag{B.29}$$

Now

$$\begin{aligned} \star\star\, (\mathrm{d} \star \alpha_p) &= (-1)^{(n-p+1)(n-n+p-1)}\, \mathrm{d} \star \alpha_p \\ \mathrm{d} \star \alpha_p &= (-1)^{(p-1)(n-p+1)} \star\star\, (\mathrm{d} \star \alpha_p) \end{aligned} \tag{B.30}$$

Using this result, we find

$$\begin{aligned} (\alpha_p, \mathrm{d}\beta_{p-1}) &= (-1)^{p+(p-1)(n-p+1)} \int_M \beta_{p-1} \wedge \star\, (\star\, \mathrm{d} \star \alpha_p) \\ &= (-1)^{n(p+1)+1} (\beta_{p-1}, \star\, \mathrm{d} \star \alpha_p) \\ &= (\delta\alpha_p, \beta_{p-1}) \end{aligned} \tag{B.31}$$

here, $p + (p-1)(n-p+1) = p + pn - p^2 + p - n + p - 1 = n(p+1) + 1 \pmod 2$ □

The adjoint derivative can be expressed as a covariant derivative.

$$(\star\, \mathrm{d} \star F)_{i_2 \dots i_p} = (-1)^{p(n-p)+p-1} \nabla^\alpha F_{\alpha i_2 \dots i_p} \tag{B.32}$$

Proof. We take

$$\star\, F_{i_{p+1} \dots i_n} = \frac{1}{p!} \varepsilon_{i_{p+1} \dots i_n j_1 \dots j_p} F_{j_1 \dots j_p} \tag{B.33}$$

Now, if we take the exterior derivative

$$\begin{aligned}(\mathrm{d}\star F)_{\alpha i_{p+1}\cdots i_n} &= (n-p+1)\partial_{[\alpha}(\star F)_{i_{p+1}\cdots i_n]} \\ &= \frac{n-p+1}{p!}\varepsilon_{i_{p+1}\cdots i_n j_1\cdots j_p}\partial_{[\alpha}F_{j_1\cdots j_p]}\end{aligned} \tag{B.34}$$

Now taking the Hodge star again

$$\begin{aligned}(\star\mathrm{d}\star F)_{i_1\ldots i_{p-1}} &= \frac{1}{(n-p+1)!}\varepsilon_{i_1\ldots i_{p-1}\alpha i_{p+1}\ldots i_n}(\mathrm{d}\star F)_{\alpha i_{p+1}\cdots i_n} \\ &= \frac{n-p+1}{p!(n+p)!}\varepsilon_{i_1\ldots i_{p-1}\alpha i_{p+1}\ldots i_n}\varepsilon_{i_{p+1}\ldots i_n j_1\ldots j_p}\partial_\alpha F_{j_1\ldots j_p} \\ &= \frac{1}{p!(n-p)!}(-1)^{p(n-p)}\varepsilon_{i_{p+1}\ldots i_n i_1\ldots i_{p-1}\alpha}\varepsilon_{i_{p+1}\ldots i_n j_1\ldots j_p}\partial_\alpha F_{j_1\ldots j_p} \\ &= \frac{1}{p!(n-p)!}(-1)^{p(n-p)}(-1)^{p-1}\varepsilon_{i_{p+1}\ldots i_n \alpha i_1\ldots i_{p-1}}\varepsilon_{i_{p+1}\ldots i_n j_1\ldots j_p}\partial_\alpha F_{j_1\ldots j_p} \\ &= \frac{1}{p!(n-p)!}(-1)^{p(n-p)+p-1}(n-p)!\delta_{j_1\ldots j_p\alpha i_1\ldots i_{p-1}}\partial_\alpha F_{j_1\ldots j_p} \\ &= (-1)^{p(n-p)+p-1}\frac{1}{p!}p!\partial_\alpha F_{\alpha i_1\ldots i_{p-1}} \\ &= (-1)^{p(n-p)+p-1}\partial_\alpha F_{\alpha i_1\ldots i_{p-1}}\end{aligned} \tag{B.35}$$

□

Equivalently, we can write

$$\delta F_{i_2\ldots i_n} = (-1)^n\nabla^\alpha F_{\alpha i_2\ldots i_n} \tag{B.36}$$

Action of a p-form

We can write the action of a p-form

$$S = \int_M F\wedge\star F \tag{B.37}$$

We define the field strength p-form

$$F_p = \mathrm{d}A_{p-1} \tag{B.38}$$

We get the Bianchi identity

$$\mathrm{d}F_p = \mathrm{d}\,\mathrm{d}A_{p-1} = 0 \tag{B.39}$$

Now, if we vary it

$$\delta F = \mathrm{d}(\delta A) \tag{B.40}$$

Varying the action gives us

$$\delta S = \int_M \delta F\wedge\star F = \int_M \mathrm{d}(\delta A\wedge\star F) - (-1)^{p-1}\,\delta A\wedge\mathrm{d}(\star F) = \int_M \delta A\wedge[(-1)^p\,\mathrm{d}\star F] \tag{B.41}$$

From least action principle

$$\delta S = 0 \Rightarrow (-1)^p\,\mathrm{d}\star F = 0 \tag{B.42}$$

Hence, the equation of motions

$$\mathrm{d}\star F = 0 \tag{B.43}$$

We can write the Einstein-Hilbert action as

$$S = \int_{\mathcal{M}} R\star 1 \tag{B.44}$$

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