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A Review of the Kazhdan–Lusztig Theory (1979)

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1. Introduction

The Kazhdan-Lusztig theory is an important topic in representation theory that links algebra, geometry and combinatorics. Introduced by David Kazhdan and George Lusztig in 1979 in their publication titled “*Representations of Coxeter Groups and Hecke Algebras*”, the theory centers on the construction of a unique basis of the Hecke Algebra called the Kazhdan-Lusztig basis. We then see how the Hecke Algebra acts on this basis using W -graphs, which encodes the representation theory of the Hecke Algebra.

In this report, we discuss the three main theorems from “*Representations of Coxeter Groups and Hecke Algebras*”. The first theorem proves the uniqueness and existence of the Kazhdan-Lusztig basis. In their paper, Kazhdan-Lusztig mention two equivalent definitions of the Kazhdan-Lusztig basis. Since their proof of Theorem 1 focuses on the original definition, we have chosen to re-prove it using the alternative definition. Theorem 2 was proved essentially the same as the authors, with further explanations where deemed necessary. Although this report does not include a proof of Theorem 3, it provides an detailed example to illustrate how it works.

We begin by reviewing the necessary preliminaries.

1.1. Coxeter System

A Coxeter System is a pair (W, S) where W is the *Coxeter group* and $S \subset W$ is the set of simple reflections which generate W such that:

$$W = \langle s \in S \mid s^2 = 1, (st)^{m(s,t)} = 1; \forall s, t \in S \rangle$$

where $m(t, s) \in \{2, 3, 2, \dots\} \cup \{\infty\}$ denotes the order of $st \in W$.

Example 1 (Symmetric Group) The *Symmetric Group* S_n is generated by the set of finite simple reflections $S = \{s_i = (i, i + 1) \mid i \in [n - 1]\}$ where each s_i is a transposition that swaps the i -th and $(i + 1)$ -th positions. Then :

$$S_n = \langle s_1, s_2, \dots, s_{n-1} \mid (s_i s_j)^{m(i,j)} = 1; i, j \in [n - 1] \rangle$$

where:

$$m(i, j) = \begin{cases} 1, & i = j \\ 3, & |i - j| = 1 \\ 2, & |i - j| \geq 2 \end{cases}$$

1.2. Bruhat Order

Let (W, S) be a Coxeter system. The Bruhat order on W is the partial order $<$ generated by the following relation:

for $x, y \in W$, we say $x < y$ if there exists a reflection $s \in S$ such that $y = xs$ and $l(y) = l(x) + 1$, where $l : W \rightarrow \mathbb{N}$ is the length function defined by $l(w) = \min\{k \in \mathbb{N} \mid w = s_1 s_2 \dots s_k; s_i \in S\}$ That is, $l(w)$ is the minimal number of simple reflections needed to express w .

1.3. Hecke Algebra:

For any Coxeter group (W, S) , we define the Hecke algebra \mathcal{H} that is a unital associative algebra over the ring $\mathbb{Z}[q, q^{-1}]$. \mathcal{H} has the standard basis $\{T_w \mid w \in W\}$ and is generated by the set $\{T_s \mid s \in S\}$ and is subjected to the following multiplication rules:

$$\begin{aligned}
\text{(i)} \quad T_w T_{w'} &= \begin{cases} T_{ww'} & , \text{if } l(ww') = l(w) + l(w') \\ T_{ww'} + (q - q^{-1})T_w & , \text{otherwise} \end{cases} \\
\text{(ii)} \quad (T_s + 1)(T_s - q) &= 0
\end{aligned} \tag{1}$$

where $l(w)$ is the length of w .

Remark: We can rearrange (1.2) to obtain the quadratic relation: $T_s^2 = qT_s - T_s + q$

Multiplying both sides by T_s^{-1} we get

$$T_s^{-1} = q^{-1}T_s + (q^{-1} + 1) \tag{2}$$

Hence, each generator T_s is invertible and consequently every $T_w, \forall w \in W$, is also invertible in \mathcal{H} .

Notation: Everytime we mention $x < y$ for any $x, y \in W$, we will be referring to the Bruhat order relation.

1.4. Representation Theory

A *Representation* of a group G on a vector space V (over a field \mathbb{F}) is a homomorphism $\rho : G \rightarrow \text{GL}(V)$, where $\text{GL}(V)$ is the group of invertible linear transformations of V . The dimension of V is called the *degree* of the representation.

Let A be an algebra over a field \mathbb{F} . A representation of A on a vector space V (over \mathbb{F}) is an algebra homomorphism $\rho : A \rightarrow \text{End}(V)$, where $\text{End}(V)$ denotes the algebra of all linear endomorphisms of V .

Equivalently, a representation of A is a structure that makes V into a (left) A -module, where the action of $a \in A$ on $v \in V$ is given by $a \cdot v = \rho(a)(v)$.

Example 2: The Symmetric group of order 3 has 6 elements: $S_3 = \{\text{id}, (12), (13), (23), (123), (132)\}$ and 3 conjugacy classes: $[1], [(12), (23), (13)], [(123), (132)]$

S_3 has 3 irreducible representations illustrated by the following character table:

S_3	id=(1)=(2)=(3)	(12),(23),(13)	(123),(132)
Trivial Representation	1	1	1
Sign Representation	1	-1	1
Standard Representation	2	0	-1

2. Kazhdan-Lusztig Basis:

Let $q \rightarrow \bar{q}$ be an *involution* on the extended ring $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$, defined by $\overline{q^{\frac{1}{2}}} = q^{-\frac{1}{2}}$. This extends to an *involution* of the ring \mathcal{H} , defined by:

$$\overline{\sum q_w T_w} = \sum \bar{q}_w T_{w^{-1}} \tag{3}$$

where $q_w = q^{l(w)}$.

A *Kazhdan-Lusztig* basis is a set $\{C_w \mid w \in W\}$ that satisfies the following properties:

$$(I) \overline{C_w} = C_w$$

$$(II) C_w = q_w^{-\frac{1}{2}} \sum_{y \leq w} P_{y,w} T_y,$$

where $P_{y,w}$ is a polynomial in q of degree $\leq \frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$ and $P_{w,w} = 1$.

Before getting into the existence and uniqueness proof of the *Kazhdan-Lusztig* basis, we define another polynomial $R_{x,y} \in \mathbb{Z}(q)$ by the formula:

$$T_{y^{-1}}^{-1} = \sum_x \overline{R_{x,y}} q_x^{-1} T_x \quad (4)$$

where we can compute $R_{x,y}$ by:

$$R_{x,y} = \begin{cases} R_{sx, sy} & , \text{ if } sx < x \text{ and } sy < y \\ R_{xs, ys} & , \text{ if } xs < x \text{ and } ys < y \end{cases} \quad (5)$$

$$R_{x,y} = (q-1)R_{sx,y} + qR_{sx, sy}, \text{ if } sx > x \text{ and } sy < y$$

Proposition 1: For all $x, y \in W$, one has $R_{x,y} \neq 0$ iff $x \leq y$ (Bruhat order).

Proof: We argue by induction on $l(y)$.

Base case: For $y = e$ (the identity), the defining properties of the R -polynomials give $T_{e^{-1}}^{-1} = \sum_x \overline{R_{x,e}} q_x^{-1} T_x = T_e$ so $R_{e,e} = 1$ and $R_{x,e} = 0$ for $x \neq e$. Hence $R_{x,e} \neq 0$ iff $x \leq e$.

Inductive step: Assuming the statement holds for all pairs (u, v) with $l(v) < l(y)$.

If $sx < x$, then by (5.1), $R_{x,y} = R_{sx, sy}$, and by the induction hypothesis $R_{sx, sy} \neq 0$ iff $sx \leq sy$ iff $x \leq y$. If $sx > x$, then $R_{x,y} = (q-1)R_{x, sy} + qR_{sx, sy}$. By induction, $R_{x, sy} \neq 0$ iff $x \leq sy$ and $R_{sx, sy} \neq 0$ iff $sx \leq sy$. Since $sy < y$, each nonzero term occurs exactly when $x \leq y$, hence the sum is nonzero iff $x \leq y$.

Combining the two cases yields $R_{x,y} \neq 0$ iff $x \leq y$. □

Theorem 1 (Existence and Uniqueness) For every Coxeter system (W, S) , there exists a unique *Kazhdan-Lusztig* basis $\{C_w : w \in W\}$ of the Hecke algebra \mathcal{H} .

Proof: (Uniqueness) We can write the equality $\overline{C_w} = C_w$ in the form

$$\begin{aligned} q_w^{-\frac{1}{2}} \sum_{y \leq w} \overline{P_{y,w}} T_y &= q_w^{-\frac{1}{2}} \sum_{y \leq w} P_{y,w} T_y \\ \Rightarrow q_w^{\frac{1}{2}} \sum_{y \leq w} \overline{P_{y,w}} T_y^{-1} &= q_w^{-\frac{1}{2}} \sum_{y \leq w} P_{y,w} T_y \\ \Rightarrow q_w^{\frac{1}{2}} \sum_{y \leq w} \overline{P_{y,w}} \sum_{y \leq x} \overline{R_{y,x}} q_y^{-1} T_y &= q_w^{-\frac{1}{2}} \sum_{y \leq w} P_{y,w} T_y \end{aligned}$$

matching the coefficients of $T_y : \forall y \leq w$,

$$\begin{aligned} q_w^{\frac{1}{2}} \sum_{y < x \leq w} \overline{P_{y,w}} \overline{R_{y,x}} q_y^{-1} &= q_w^{-\frac{1}{2}} P_{y,w} \\ \Rightarrow q_w^{\frac{1}{2}} q_x^{-1} \overline{P_{x,w}} \overline{R_{x,x}} + q_w^{\frac{1}{2}} \sum_{y < x \leq w} \overline{P_{y,w}} \overline{R_{y,x}} q_y^{-1} &= q_w^{-\frac{1}{2}} P_{y,w} \\ \Rightarrow q_w^{\frac{1}{2}} \sum_{y < x \leq w} \overline{P_{y,w}} \overline{R_{y,x}} q_y^{-1} &= q_w^{-\frac{1}{2}} P_{y,w} - q_w^{\frac{1}{2}} q_x^{-1} \overline{P_{x,w}} \end{aligned}$$

Then $\forall y < w$,

$$q_w^{\frac{1}{2}} \sum_{y < x \leq w} \overline{P_{y,w}} \overline{R_{y,x}} q_y^{-1} = q_w^{-\frac{1}{2}} P_{y,w} - q_w^{\frac{1}{2}} q_y^{-1} \overline{P_{y,w}}$$

multiplying with $q^{\frac{1}{2}}$ on both sides,

$$q_w^{\frac{1}{2}} q_y^{-\frac{1}{2}} \sum_{y < x \leq w} \overline{P_{y,w}} \overline{R_{y,x}} = q_w^{-\frac{1}{2}} q_y^{\frac{1}{2}} P_{y,w} - q_w^{\frac{1}{2}} q_y^{-\frac{1}{2}} \overline{P_{y,w}} \quad (6)$$

In R.H.S, we have $q_w^{-\frac{1}{2}} q_y^{\frac{1}{2}} P_{y,w}$ and $q_w^{\frac{1}{2}} q_y^{-\frac{1}{2}} \overline{P_{y,w}}$

$P_{y,w} \in \mathbb{Z}[q]$ with $w > y$, so $q_w^{-\frac{1}{2}} q_y^{\frac{1}{2}} P_{y,w}$ is a polynomial in $q^{-\frac{1}{2}}$, without a constant term.

$\overline{P_{y,w}} \in \mathbb{Z}[q]$ with $w > y$, so $q_w^{\frac{1}{2}} q_y^{-\frac{1}{2}} \overline{P_{y,w}}$ is a polynomial in $q^{\frac{1}{2}}$, without a constant term.

Thus, there can never be cancellations between them.

Now, assuming there is another basis $\{C_w'\}$, then $C_w' = q_w^{-\frac{1}{2}} \sum_{y \leq w} P_{y,w} T_y$ and it would satisfy (5) for $P_{y,w}'$ and $\overline{P_{y,w}'}$, i.e. we get

$$q_w^{\frac{1}{2}} q_y^{-\frac{1}{2}} \sum_{y < x \leq w} \overline{P_{y,w}'} \overline{R_{(y,x)'}} = q_w^{-\frac{1}{2}} q_y^{\frac{1}{2}} P_{y,w}' - q_w^{\frac{1}{2}} q_y^{-\frac{1}{2}} \overline{P_{y,w}'} \quad (7)$$

If we subtract (7) with (6) and write $\Delta = P_{y,w} - P_{y,w}'$, the L.H.S. cancel out and we are left with:

$$q_w^{-\frac{1}{2}} q_y^{\frac{1}{2}} \Delta - q_w^{\frac{1}{2}} q_y^{-\frac{1}{2}} \overline{\Delta} = 0 \quad (8)$$

But as discussed earlier, these terms can never cancel each other out. Hence, $C_w' = C_w$. This proves the uniqueness of the Kazhdan-Lusztig basis.

(Existence) We will prove this using induction. For the base case, $C_e = T_e$. For the induction step, assume we have already constructed C_v for all $v < w$.

Then, $C_s = q_s^{-\frac{1}{2}} \sum_{v \leq s} P_{v,s} T_v = q^{-\frac{1}{2}} (1 + T_s)$

For $sw < w$, C_{sw} is already constructed:

$$C_{sw} = q_{sw}^{-\frac{1}{2}} \sum_{v \leq sw} P_{v,sw} T_v$$

$$C_s C_{sw} = (q^{-\frac{1}{2}} (1 + T_s)) q_{sw}^{-\frac{1}{2}} \sum_{v \leq sw} P_{v,sw} T_v$$

$$= q^{-\frac{1}{2}(l(sw)+1)} \sum_{v \leq sw} P_{v,sw} (1 + T_s) T_v$$

$$= q_w^{-\frac{1}{2}} \sum_{v \leq sw} P_{v,sw} T_v + q_w^{-\frac{1}{2}} \sum_{v \leq sw} P_{v,sw} T_s T_v$$

$$= q_w^{-\frac{1}{2}} \sum_{v \leq sw} P_{v,sw} T_v + q_w^{-\frac{1}{2}} \sum_{v \leq sw} P_{v,sw} T_{sv} + q_w^{-\frac{1}{2}} \sum_{v \leq sw, sv < v} P_{v,sw} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) T_v$$

$$= q_w^{-\frac{1}{2}} \sum_{v \leq sw} P_{v,sw} T_v + q_w^{-\frac{1}{2}} \sum_{v \leq sw} P_{v,sw} (1 + q^{\frac{1}{2}} - q^{-\frac{1}{2}}) T_v$$

$(1 + q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \notin P_{v,sw} \in \mathbb{Z}[q]$. $P_{v,sw}$ has a degree bound $\leq \frac{1}{2}(l(sw) - l(v) - 1)$.

$q^{-\frac{1}{2}}$ multiplies with $P_{v,sw}$ and lowers it's degree by $\frac{1}{2}$, keeping it within the degree bound. But $q^{+\frac{1}{2}}$ multiplies to give the top degree of exactly $\frac{1}{2}(l(sw) - l(v) - 1)$. This violates the rule $P_{v,sw} \in \mathbb{Z}[q]$.

Hence, we must find a way to cancel this term out.

We define

$$\mu(v, sw) := \text{coefficient of } q^{\frac{1}{2}(l(sw)-l(v)-1)}, \quad (9)$$

in $P_{v,sw}$.

$$q^{\frac{1}{2}} \mu(v, sw) q^{\frac{1}{2}(l(sw)-l(v)-1)} = \mu(v, sw) q^{\frac{1}{2}(l(w)-l(v)-1)}$$

So there is a $\mu(v, sw)q^{\frac{1}{2}(l(w)-l(v)-1)}$ term in $C_s C_{sw}$ where $sv < v$. This is precisely the term we want to cancel out.

$$\begin{aligned}
\mu(v, sw)C_v &= \mu(v, sw)q_v^{-\frac{1}{2}} \sum_{y \leq v} P_{y,v} T_y \\
&= q^{-\frac{1}{2}(l(w))} \mu(v, sw) q^{\frac{1}{2}} \sum_{y \leq v} P_{y,v} T_y \\
&= q_w^{-\frac{1}{2}} \mu(v, sw) q^{\frac{1}{2}} \left(q^{\frac{1}{2}(l(v)-l(v)-1)} \right) T_v + \text{lower order terms} \\
&= q_w^{-\frac{1}{2}} \mu(v, sw) \left(q^{\frac{1}{2}(l(v)+1-l(v)-1)} \right) T_v + \text{lower order terms} \\
&= q_w^{-\frac{1}{2}} \mu(v, sw) \left(q^{\frac{1}{2}(l(w)-l(v)-1)} \right) T_v + \text{lower order terms}
\end{aligned}$$

\therefore There is also a $\mu(v, sw)q^{\frac{1}{2}(l(w)-l(v)-1)}$ term in $\mu(v, sw)C_v \forall v < w$.

We define $C_w := C_s C_{sw} - \sum_{v < w} \mu(v, sw)C_v$, to cancel out the unwanted $q^{\frac{1}{2}}$ terms. This leaves $C_w = q_w^{-\frac{1}{2}} \sum_{v \leq w} P_{v,w} T_v$.

To check for the degree bound, $C_w = q_w^{-\frac{1}{2}} \sum_{v \leq sw} P_{v,sw} T_v + q_w^{-\frac{1}{2}} \sum_{v \leq sw} P_{v,sw} (1 - q^{-\frac{1}{2}}) T_v$

$\deg(P_{v,sw}) \leq \frac{1}{2}(l(sw) - l(v) - 1) < \frac{1}{2}(l(w) - l(v) - 1)$ and

$$\begin{aligned}
\deg(q^{-\frac{1}{2}} P_{v,sw}) &\leq \frac{1}{2}(l(sw) - l(v) - 1) - \frac{1}{2} \\
&= \frac{1}{2}(l(w) - l(v) - 3) \\
&< \frac{1}{2}(l(w) - l(v) - 1)
\end{aligned}$$

$\therefore \deg(P_{v,w}) \leq \frac{1}{2}(l(w) - l(v) - 1)$ for $v < w$.

This concludes are property (II).

Checking property (I),

$$\begin{aligned}
\overline{C_w} &= \overline{C_s C_{sw} - \sum_{v < w} \mu(v, sw)C_v} \\
&= C_s C_{sw} - \sum_{v < w} \mu(v, sw)C_v \text{ [by induction, all } v < w \text{ satisfy (I) property]}
\end{aligned}$$

Hence, $\overline{C_w} = C_w$. This concludes the existence proof of the Kazhdan-Lusztig basis. □

Remark: The degree bound is necessary because without it there would be infinitely many possible KL bases which would destroy the uniqueness.

Suppose we define $C_w' = C_w + \sum_{y < w} q_w^{-\frac{1}{2}} f(q) C_y$, where $f(q) \in \mathbb{Z}[q]$ with $\deg(f(q)) > \frac{1}{2}(l(w) - l(v) - 1)$. This new C_w' is still bar-invariant and triangular but without the degree bound there are infinitely many choices of $f(q)$.

3. W-Graphs:

Let (W, S) be a Coxeter System. A *W-Graph* consists of:

- (a) a set of vertices X ,
- (b) each vertex $x_1 \in X$ is labelled by a subset $I_x \subset S$,
- (c) a set of edges Y between two vertices $x_1, x_2 \in X$ weighted by an integer $\mu(x_1, x_2) \neq 0$, if there is an edge between them.

Let E be the free $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -module with basis $\{x : x \in X\}$. We define an action of the Hecke algebra \mathcal{H} on E by

$$T_s(x) = \begin{cases} -x & , s \in I_x \\ qx + q^{\frac{1}{2}} \sum_{\{y \in X, s \in I_y\}} \mu(y, x)y & , s \notin I_x \end{cases} \quad (10)$$

and for any $s \neq t$ in S such that $m(st) = m$,

$$\underbrace{T_s T_t T_s \dots}_{m \text{ factors}} = \underbrace{T_t T_s T_t \dots}_{m \text{ factors}} \quad (11)$$

This action extends to a unique representation $\varphi : \mathcal{H} \rightarrow \text{End}(E)$.

Example 3: Let $W = S_3$

– Vertices: elements of $S_3 = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$

– Labels: $I_s = \{s \in S \mid sw < w\}$

- $I_e = \emptyset$
- $I_{\{s_1\}} = \{s_1\}$, since $s_1 s_1 = e < s_1$
- $I_{\{s_2\}} = \{s_2\}$
- $I_{\{s_1 s_2\}} = \{s_2\}$, since $s_2(s_1 s_2) = s_1 < s_1 s_2$
- $I_{\{s_2 s_1\}} = \{s_1\}$
- $I_{\{s_1 s_2 s_1\}} = \{s_1, s_2\}$

– Edges: between $\{s, t\} \in S$ if $s < t$, with $\mu(s, t) = 1$ for every edge.

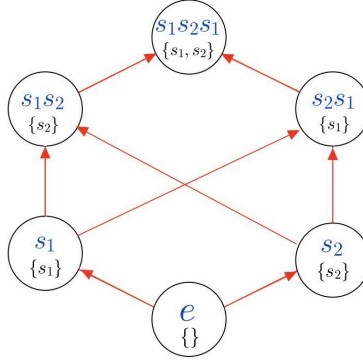


Figure 1: W -graph of S_3 group.

Given $y, w \in W$, we define \prec by saying that $y \prec w$ if the following conditions are satisfied:

- (i) $y < w$,
- (ii) $(-1)^{l(y)} = -(-1)^{l(w)}$,
- (ii) $\deg(P_{y,w}) = \frac{1}{2}(l(w) - l(y) - 1)$, exactly.

We again denote $\mu(y, w) :=$ coefficient of the highest power of q in $P_{y,w}$. It is a non-zero integer. If $w \prec y$, we set $\mu(w, y) = \mu(y, w)$.

We now define W^0 to be the group opposed to W . This means W^0 is simply a copy of the group W generated by the simple reflections $S^0 = \{s^0 \mid s \in S\}$. $W^0 \cong W$ as Coxeter groups and S^0 is a disjoint copy of S .

So, essentially (W^0, S^0) is the same group as (W, S) but we label it as *opposed* to keep it as a separate copy, to keep track of left descents and right descents.

Then $(W \times W^0, S \sqcup S^0)$ is also a Coxeter group, because products of Coxeter groups form Coxeter groups.

Let Γ_W be the graph whose vertices are elements of W and edges are between pairs $\{y, w\}$ whenever $y \prec w$, meaning edges only exist between pairs where one is immediately below the other.

For each $w \in W$, let $I_w = \mathcal{L}(w) \sqcup \mathcal{R}(w)^0 \subset S \sqcup S^0$ where:

$$\mathcal{L}(w) = \{s \in S \mid sw < w\}$$

$$\mathcal{R}(w) = \{s \in S \mid ws < w\}$$

$$\mathcal{R}(w)^0 = \{s^0 \mid s \in \mathcal{R}(w)\}$$

Theorem 2: Γ_W , together with the assignment $w \rightarrow I_w$ and with the function μ defined above, is a $W \times W^0$ -graph.

Proof: We want to show that (Γ_W, I_w, μ) is a $W \times W^0$ -graph, i.e. it satisfies the following conditions:

(i)

$$T_s C_y = \begin{cases} -C_y & , \text{if } s \in \mathcal{L}(y) = \{s \mid sy < y\} \\ qC_y + q^{\frac{1}{2}} \sum_{\{x < y, s \in \mathcal{L}(x), sx < x\}} \mu(y, x) C_x & , \text{if } s \notin \mathcal{L}(y), sy > y \end{cases} \quad (12)$$

(ii)

$$C_y T_s = \begin{cases} -C_y & , \text{if } s \in \mathcal{R}(y) = \{s \mid ys < y\} \\ qC_y + q^{\frac{1}{2}} \sum_{\{x < y, s \in \mathcal{R}(x), xs < x\}} \mu(y, x) C_x & , \text{if } s \notin \mathcal{R}(y), ys > y \end{cases} \quad (13)$$

In their proof of Theorem 1, the authors use the following alternative definition of the Kazhdan-Lusztig basis:

$$C_w = \sum_{x \leq w} \varepsilon_x \varepsilon_w q_w^{\frac{1}{2}} q_x^{-1} \overline{P_{x,y}} T_x, \text{ where } \varepsilon_x = (-1)^{l(x)} \quad (14)$$

and define C_w for $w = sv, sv > v$ as:

$$C_w := \left(q^{-\frac{1}{2}} T_s - q^{\frac{1}{2}} \right) C_v - \sum_{\{z \prec v, sz < z\}} \mu(z, v) C_z, \text{ if } s \in S \text{ and } sv = w \quad (15)$$

replacing $w \rightarrow sv, sv \rightarrow v$ in (15),

$$C_{sv} := \left(q^{-\frac{1}{2}} T_s - q^{\frac{1}{2}} \right) C_v - \sum_{\{z \prec v, sz < z\}} \mu(z, v) C_z,$$

which we can rearrange to get:

$$T_s C_v = qC_v + q^{\frac{1}{2}} C_{sv} + q^{\frac{1}{2}} \sum_{\{z \prec v, sz < z\}} \mu(z, v) C_z, \text{ if } s \in \mathcal{L}(y) \quad (16)$$

replacing $sv \leftrightarrow v$, i.e. $sv < v$,

$$\begin{aligned}
C_v &= \left(q^{-\frac{1}{2}}T_s - q^{\frac{1}{2}}\right)C_{sv} - \sum_{\{z \prec sv, sz < z\}} \mu(z, sv)C_z \\
T_s C_v &= T_s \left(\left(q^{-\frac{1}{2}}T_s - q^{\frac{1}{2}}\right)C_{sv} - \sum_{\{z \prec sv, sz < z\}} \mu(z, sv)C_z \right) \\
&= q^{-\frac{1}{2}}T_s^2 C_{sv} - q^{\frac{1}{2}}T_s C_{sv} - \sum_{\{z \prec sv, sz < z\}} \mu(z, sv)T_s C_z \\
&= q^{-\frac{1}{2}}(qT_s - T_s + q)C_{sv} - q^{\frac{1}{2}}T_s C_{sv} + \sum_{\{z \prec sv, sz < z\}} \mu(z, sv)C_z \quad [T_s C_z = -C_z, sz < z] \\
&= -q^{-\frac{1}{2}}T_s C_{sv} + q^{\frac{1}{2}}C_{sv} + \sum_{\{z \prec sv, sz < z\}} \mu(z, sv)C_z \\
&= -\left[\left(q^{-\frac{1}{2}}T_s - q^{\frac{1}{2}}\right)C_{sv} - \sum_{\{z \prec sv, sz < z\}} \mu(z, sv)C_z \right] \\
&= -C_v
\end{aligned}$$

$\therefore T_s C_v = -C_v$, if $sv > v, s \notin \mathcal{L}(y)$.

Hence, Γ_W satisfies condition (i).

Similarly, if we interchange left multiplication in the definition of C_w to right multiplication:

$$C_{vs} := C_v \left(q^{-\frac{1}{2}}T_s - q^{\frac{1}{2}} \right) - \sum_{\{z \prec v, zs < z\}} \mu(z, v)C_z, \text{ if } s \in S \text{ and } vs > v$$

we can rearrange this to get,

$$C_v T_s = qC_v + q^{\frac{1}{2}}C_{sv} + q^{\frac{1}{2}} \sum_{\{z \prec v, sz < z\}} \mu(z, v)C_z, \text{ if } s \in \mathcal{R}(y) \quad (17)$$

Again replacing $vs \leftrightarrow v$, i.e. $vs < v$, will give us $C_v T_s = -C_v$ when $s \notin \mathcal{R}(y)$.

So, Γ_W also satisfies condition (ii).

We lastly check that μ is a nonzero integer.

Claim (a) : Let $x, y \in W, s \in S$ be such that $x < y, sy < y, sx > x$. Then $x \prec y$ if and only if $y = sx$. (b) Let $x, y \in W, s \in S$ be such that $x < y, ys < y, xs > x$. Then $x \prec y$ if and only if $y = xs$.

Proof: (a) For $x < y$ and $sy < y$,

$$\begin{aligned}
T_s C_y &= -C_y \\
T_s \left(q_y^{-\frac{1}{2}} \sum_{x \leq y} P_{x,y} T_x \right) &= - \left(q_y^{-\frac{1}{2}} \sum_{x \leq y} P_{x,y} T_x \right) \\
q_y^{-\frac{1}{2}} \sum_{x \leq y} P_{x,y} T_s T_x &= -q_y^{-\frac{1}{2}} \sum_{x \leq y} P_{x,y} T_x
\end{aligned}$$

for $sx > x$,

$$q_y^{-\frac{1}{2}} \sum_{x \leq y} P_{x,y} T_{sx} = -q_y^{-\frac{1}{2}} \sum_{x \leq y} P_{x,y} T_x$$

$$\text{for } T_{sx} : q_y^{-\frac{1}{2}} P_{x,y} = -q_y^{-\frac{1}{2}} P_{sx,y}$$

$$\Rightarrow P_{x,y} = -P_{sx,y}$$

$$\therefore \deg(P_{x,y}) = \deg(P_{sx,y})$$

$$\begin{aligned}
\text{If } y \neq sx, \deg(P_{x,y}) = \deg(P_{sx,y}) &\leq \frac{1}{2}(l(y) - l(sx) - 1) \\
&= \frac{1}{2}(l(y) - l(x) - 2) \\
&< \frac{1}{2}(l(y) - l(x) - 1)
\end{aligned}$$

Hence, if $y \neq sx$, the relation $x \prec y$ is not satisfied.

$$\therefore y = sx \text{ and } P_{x,y} = -P_{sx,y} = -P_{y,y} = -1, \text{ i.e. } \mu(x, y) = -1.$$

(b) We can prove the right multiplication case similarly:

For $x < y, xs > x$ and $ys < y$,

$$C_y T_s = -C_y$$

$$\Rightarrow P_{x,y} = -P_{xs,y}$$

$$\text{If } y \neq xs, \deg(P_{x,y}) = \deg(P_{xs,y}) < \frac{1}{2}(l(y) - l(x) - 1)$$

Hence, $x < y$ if and only if $y = xs$, and similarly $\mu(x, y) = -1$.

$\therefore \mu$ is a non-zero integer and (Γ_W, I_w, μ) is a $W \times W^0$ -graph. □

Remark: Claim (a) and (b) follow statements (2.3e) and (2.3f) from the original paper. However, we believe the original statement in (2.3f) includes an error where it says “Let $x, y \in W, s \in S$ be such that $x < y, ys < y, xs < x$. Then $x < y$ if and only if $y = xs$ ”. Here, if we we have $xs < x$ then we cannot have $y = xs$ because that would imply $xs < x < y$. Hence, in our statement (b) we have changed $xs < x$ to $xs > x$ which follows directly from (a) for the right multiplication case.

4. Left cell representation:

In a W -graph, we say $x_0 \sim x_1$ if $\mu(x_0, x_1) \neq 0$ and $x \leq_L x'$ if there is a path $x = x_0 \sim x_1 \sim \dots \sim x_i = x'$ such that:

- each $\{x_{i-1}, x_i\}$ is an edge in the W -graph, and
- $I_{x_{i-1}} \not\subseteq I_{x_i}$

In simple words, $x \leq_L x'$ if there is a directed path from x to x' . We write $x \sim_L x'$ if $x \leq_L x'$ and $x' \leq_L x$. Each equivalence class of this relation forms a connected piece of the graph that we call a left cell.

Example 4: Looking back at our S_3 -graph, the left cells are:

- $\{e\}$
- $\{s_1, s_1 s_2\}$
- $\{s_2, s_2 s_1\}$
- $\{s_1 s_2 s_1\}$

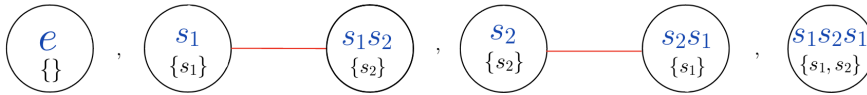


Figure 2: Left cells of the S_3 group.

Theorem 3: Let X be a left cell of $W = S_n$, let Γ be the W -graph associated to X and let ρ be the representation of \mathcal{H} (over the quotient field of $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$) corresponding to Γ . Then ρ is irreducible and the isomorphism class of the W -graph Γ depends only on the isomorphism class of ρ and not on X .

Explanation: Given X is a left cell, then each left cell gives a subspace of the Hecke Algebra (spanned by its basis elements C_x for $x \in X$) that is stable under left multiplication. Hence, each left cell gives rise to one such W -graph.

$\rho : \mathcal{H} \rightarrow \text{End}(V)$ is the cell representation associated to that X .

According to theorem 3, this representation is:

- (i) Irreducible: the cell subspace cannot be broken into smaller invariant subspaces
- (ii) Independent of X : if two left cells X and X' happen to yield isomorphic representations $\rho \cong \rho'$ then the W -graphs are also isomorphic as graphs.

We unpack this with our final example.

Example 4: S_3 has four left cells: $X_1 = \{e\}$, $X_2 = \{s_1, s_1 s_2\}$, $X_3 = \{s_2, s_2 s_1\}$, $X_4 = \{s_1 s_2 s_1\}$

Each cell X_i gives a subrepresentation V_i of the Hecke Algebra, where V_i has basis $\{b_x \mid x \in X\}$ subjected to:

$$T_s(b_x) = \begin{cases} -b_x & , s \in I_x \\ qb_x + q^{\frac{1}{2}} \sum_{\{y \in X, s \in I_y\}} \mu(y, x) b_y & , s \notin I_x \end{cases} \quad (18)$$

(i) $X_1 = \{e\}$: The subrepresentation V_1 has only one basis element $\{b_e\}$

Since $s_1, s_2 \notin I_e$, $T_s b_e = qb_e$, so both T_{s_1} and T_{s_2} act on the basis element by multiplication with q . This is precisely the *trivial representation* with dimension 1, and it is irreducible.

(ii) $X_4 = \{s_1 s_2 s_1\}$: The subrepresentation V_4 also has only one basis element $\{b_{s_1 s_2 s_1}\}$ and $I_{s_1 s_2 s_1} = \{s_1, s_2\}$.

Both $s_1, s_2 \in I_{s_1 s_2 s_1}$, and $T_s b_{s_1 s_2 s_1} = -b_{s_1 s_2 s_1}$. This is precisely the *sign representation* with dimension 1, and it is irreducible.

(iii) $X_2 = \{s_1, s_1 s_2\}$: The subrepresentation V_2 has two basis elements $\{b_{s_1}, b_{s_1 s_2}\}$ and $I_{s_1} = \{s_1\}$ and $I_{s_1 s_2} = \{s_2\}$.

$$\begin{aligned} T_1 b_{s_1} &= -b_1 \\ T_1 b_{s_1 s_2} &= qb_{s_1 s_2} + b_{s_1} \\ T_2 b_{s_1} &= qb_{s_1} + b_{s_1 s_2} \\ T_2 b_{s_1 s_2} &= -b_{s_2} \end{aligned}$$

In matrix form this is: $\rho_{X_2}(T_1) = \begin{pmatrix} -1 & 1 \\ 0 & q \end{pmatrix}$ $\rho_{X_2}(T_2) = \begin{pmatrix} q & 0 \\ 1 & -1 \end{pmatrix}$

at $q = 1$ this is:

$$\rho_{X_2}(T_1) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \quad \rho_{X_2}(T_2) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \quad (19)$$

This is exactly the *standard representation* with dimension 2 and is irreducible.

(iv) $X_3 = \{s_2, s_2 s_1\}$: The subrepresentation V_3 also has two basis elements $\{b_{s_2}, b_{s_2 s_1}\}$ and $I_{s_2} = \{s_2\}$ and $I_{s_2 s_1} = \{s_1\}$.

And similarly, we get the representation:

$$\rho_{X_3}(T_1) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \quad \rho_{X_3}(T_2) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \quad (20)$$

This is isomorphic to the representation ρ_{X_2} .

It is also clear from the structure of X_2 and X_3 in Figure 2 that their left cell graphs are isomorphic: they are mirror images under the interchange $s_1 \leftrightarrow s_2$.

Hence, we have successfully reproduced the full representation theory of S_3 by its four left cells and its W -graph, as suggested by Theorem 3.

5. Conclusion

This report explored the Kazhdan-Lusztig theory by focusing on its three main theorems and working out extensive examples. Approaching Theorem 1 with an alternative definition allowed us to gain a clearer understanding of the Kazhdan-Lusztig basis and its construction. We followed their proof of Theorem 2 closely and found an error in printing where line (2.3f) of their paper says $xs < x$ when it should actually say $xs > x$. Lastly, due to time limitations we were unable to prove Theorem 3 but were able to add a clear example that demonstrates how it works for the S_3 group. Overall, this report reviewed the work of Kazhdan and Lusztig and illustrated how the Kazhdan-Lusztig basis reveals the link between the algebra of Hecke algebras and the geometry underlying representation theory.

6. References

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Approval

The internship report titled “A Review of the Kazhdan-Lusztig Theory (1979)” submitted by

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