Approval

The internship report titled "A Formalism for Computing In–Out Correlators in de Sitter Space and Its Applications" submitted by **Tahmeed Hossain**, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfillment of the requirements of the internship program.

The internship was conducted under the supervision of Samanta Saha during the period 15 July 2025 to 15 October 2025.

I am approving an extension for the submission deadline until 26/11/2025.

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A Formalism to calculate In-Out Correlators in de Sitter Space and it's Application

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1 Introduction

Quantum Field Theories are usually used to fit particle scattering/collision experimental observation where we know both the input state and the output state statistics. The transition probability, scattering cross section etc are then found experimentally and the trends are fit with the theoretical predictions of the relevant quantum field theory. This requires us to build the S matrix, $s = \langle \text{out}|\text{in}\rangle = \langle 0|f(\phi)|0\rangle$. From this S-Matrix we find the propagation amplitude.

However, in case of cosmology and scalar fields in curved space-time like de Sitter as in our case, we don't really have access to the output state(or even input state!). However, in case of input state we can take a Bunch-Davies vacuum which is asymptotically a free vacuum in minkowskian space in far past. But, as the de Sitter background is highly interacting with the field, it is not possible to find a free vacuum by turning the interaction off adiabatically in the far future.

But in case of cosmology, there is a clever way to avoid these. In cosmology we measure the distribution(anisotropy) of mass and energy throughout the universe, anisotropy of CMB etc. In some sense, we can see this as being indirectly caused by inflaton field at some early time of the universe which got imprinted in space at some specific moment and propagated throughout the history of universe. So answering the following question is enough: What is the expectation value(probabilistically weighted) that given a Bunch-Davies initial state, we get a particular correlation of distribution of the fluctuation at the time of inflation's end? Then this correlation of fluctuation can be used to fit the observed anisotropies of a later time(present) by a transfer function. In order to tackle this, we can simply use the Schwinger-Keldysh formalism, where we will propagate the field both forward and backward in time. The state evolves from BD vacuum to the end of inflation and then the contour reverses and evolves reversibly backward in time to end in BD vacuum again. This formalism, known as in-in formalism is very common in QFT calculations in cosmology.

Recently, Donath and Pajer developed a calculation trick to calculate in-out states in de Sitter space[1] . Our work focused on understanding this methodology in detail. This is essential to figure out possible QFTs that may describe physically meaningful quantum fields in de-sitter spacetime. The main trick here was to build the out state from the BD vacuum in a contracting patch. The in state is built from the BD vacuum in expanding patch as usual. Then the authors proved how calculated in-out correlators match the in-in correlators(in both minkowskian and de-sitter space). This can be seen rather intuitively; just as we are evolving from the BD vacuum state to the correlator's time and then time propagating back to the BD vacuum again in the case of in-in calculation, in the in-out formalism we are building the bra by time reversing from BD vacuum in contracting patch. So, the unity of in-out and in-in correlator follows intuitively and expectedly.

As an example of it's application, we studied another work[2] where it was shown that using this formalism, we can calculate the S matrix and scattering amplitudes. However, the scattering amplitudes do not have to precisely follow the energy conservation as there are two parts of it, one with an energy conserving delta function and the other without any energy conserving delta function. The authors postulated from stability argument that the term without energy conserving delta must vanish. Then they calculated the propagation amplitude for a series of scalar fields. They found out the possible field theories that satisfy the constraint that the term without delta must vanish.

2 Theory

2.1 In–In and In–Out Correlation Functions in Curved Spacetime

In quantum field theory on curved backgrounds, the form of correlation functions depends crucially on the causal and asymptotic structure of the spacetime. Two distinct but related formalisms are commonly employed: the *in–out* (or scattering) approach and the *in–in* (Schwinger–Keldysh, closed-time-path) formalism. Although equivalent in stationary backgrounds with well-defined asymptotic vacua, these formalisms diverge sharply in time-dependent geometries such as expanding FLRW universes or de Sitter space. The key distinction arises from the nonexistence of a preferred "out" vacuum and the requirement that cosmological observables correspond to expectation values at finite times, rather than transition amplitudes between asymptotic states.

In Minkowski spacetime the presence of a global timelike Killing vector selects a natural positive-frequency decomposition, which yields a unique vacuum at both past and future timelike infinity. One may then define asymptotic *in* and *out* states via the adiabatic switching off of interactions,

$$|\mathrm{in}\rangle = \lim_{t\to-\infty} e^{iH_0 t}e^{-iHt}|\Psi\rangle, \qquad |\mathrm{out}\rangle = \lim_{t\to+\infty} e^{iH_0 t}e^{-iHt}|\Psi\rangle, \tag{1}$$

where H_0 denotes the free Hamiltonian and H the full interacting Hamiltonian. Scattering amplitudes and conventional perturbation theory are obtained from *in–out* correlators,

$$\langle \mathrm{out}|T\{\phi(x_1)\cdots\phi(x_n)\}|\mathrm{in}\rangle, \tag{2}$$

which encode transitions between asymptotic states and automatically produce the familiar Feynman propagator.

In generic curved spacetimes the assumptions underlying this construction break down. In an expanding FLRW universe, for example, the Hamiltonian acquires explicit time dependence through the scale factor $a(t)$, and even the modes of a free scalar field satisfy

$$\phi_k''(\eta) + \left(k^2 - \frac{a''}{a}\right)\phi_k(\eta) = 0, \tag{3}$$

where primes denote derivatives with respect to conformal time η. The term a''/a is generically nonvanishing, preventing the field from approaching free, asymptotically oscillatory behavior at late times. Furthermore, particle definitions depend on the observer, and the resulting "out" Fock space is typically not unitarily equivalent to the "in" Fock space. Consequently, the limit defining $|\mathrm{out}\rangle$ may not exist, and in–out amplitudes lose their operational meaning.

Cosmological observables—such as the primordial power spectrum and higher-point functions—are fundamentally expectation values of operators at a *finite* time on a fixed initial quantum state, usually taken to be the Bunch–Davies vacuum. These quantities are inherently probabilistic rather than transition amplitudes. The suitable framework is therefore the *in–in* formalism, in which one computes

$$\langle \Omega | \mathcal{O}(t) | \Omega \rangle, \tag{4}$$

where $|\Omega\rangle$ denotes the initial state prepared at early times. In the interaction picture this expectation value can be written as the Schwinger–Keldysh contour expression

$$\langle \Omega | \mathcal{O}(t) | \Omega \rangle = \left\langle \Omega \left| \bar{T} \exp\left(i \int_{-\infty}^{t} dt'\, H_I(t') \right) \mathcal{O}_I(t)\, T \exp\left(-i \int_{-\infty}^{t} dt'\, H_I(t') \right) \right| \Omega \right\rangle, \tag{5}$$

where T and $\bar{T}$ denote time- and anti-time-ordering along the forward and backward branches of the closed time contour. This structure ensures unitarity and produces a 2×2 matrix of propagators that encodes the full causal behavior of the theory.

In spacetimes lacking future asymptotic regions—most notably de Sitter space, where comoving observers possess cosmological horizons—the in–out formalism cannot be endowed with a consistent physical interpretation. The continual stretching of wavelengths leads to particle production at all late times, and the vacuum evolves into a highly squeezed state rather than approaching a free "out" vacuum. Accordingly, the S-matrix is not well defined in de Sitter space, and physical observables must be derived from in–in correlators computed at finite times.

In summary, in–out correlators describe scattering amplitudes between asymptotic free states and require the existence of such states, which is not guaranteed in nonstationary spacetimes. In–in correlators, by contrast, yield expectation values of operators at finite times and remain well defined in cosmological settings. The in–in formalism is therefore the natural and physically meaningful framework for quantum field theory in curved spacetime.

2.2 de Sitter spacetime and Patches

De Sitter space dS_d may be defined as the hyperboloid embedded in $R^{1,d}$ with metric $\eta_{AB} = \text{diag}(-1, 1, \ldots, 1)$:

$$-X_0^2 + \sum_{i=1}^{d} X_i^2 = H^{-2}, \qquad H^2 = \frac{\Lambda}{d-1}.$$

Different choices of coordinates correspond to different slicings of this hyperboloid, each illuminating a distinct causal or geometric structure.

Global slicing. Global coordinates (t, Ω_{d-1}) cover the full manifold:

$$ds^2 = -dt^2 + H^{-2} \cosh^2(Ht)\, d\Omega_{d-1}^2.$$

The scale factor $a(t) = H^{-1} \cosh(Ht)$ makes manifest the presence of both *expanding* and *contracting* phases.

Flat (inflationary) slicing. The spatially flat slicing describes an exponentially expanding universe:

$$ds^2 = -dt^2 + e^{2Ht} d\vec{x}^{\,2} = \frac{1}{(H\eta)^2} \left(-d\eta^2 + d\vec{x}^{\,2}\right), \qquad \eta < 0.$$

This patch covers only the expanding half of de Sitter space.

Contracting patch. A time-reversed version of the flat slicing yields a contracting region:

$$ds^2 = -dt^2 + e^{-2Ht} d\vec{x}^{\,2} = \frac{1}{(H\eta)^2} \left(-d\eta^2 + d\vec{x}^{\,2}\right), \qquad \eta > 0.$$

This describes the past half of the global hyperboloid, relevant when defining in-vacua or studying pre-inflationary dynamics.

Static patch. The causal diamond of a timelike observer is captured by the static coordinates:

$$ds^2 = -(1 - H^2 r^2)\, dt^2 + \frac{dr^2}{1 - H^2 r^2} + r^2 d\Omega_{d-2}^2.$$

A cosmological horizon appears at $r = H^{-1}$, associated with the Gibbons–Hawking temperature $T = \frac{H}{2\pi}$.

Expanding vs. contracting regions. In global coordinates,

$$a(t) = H^{-1}\cosh(Ht) = \frac{1}{2H}\left(e^{Ht} + e^{-Ht}\right),$$

so that

$$t \to +\infty: \quad \text{expanding patch,}$$
$$t \to -\infty: \quad \text{contracting patch.}$$

These regions correspond to the two asymptotic hemispheres of the de Sitter hyperboloid and are central in constructions of in/out vacua and studies of particle production via Bogoliubov transformations.

2.3 Bunch–Davies Vacuum and the Role of Expanding/Contracting Poincaré Patches

In quantum field theory on de Sitter space, the vacuum state is not uniquely determined by symmetry alone. Although de Sitter spacetime is maximally symmetric, the lack of a global timelike Killing vector obscures the definition of positive-frequency modes. The Bunch–Davies (Euclidean) vacuum provides a canonical choice[3]: it is de Sitter invariant, satisfies the Hadamard condition, and arises naturally from analytic continuation of the Euclidean manifold.

In the flat Poincaré slicing the metric is

$$ds^2 = a^2(\eta)\left(-d\eta^2 + d\mathbf{x}^2\right), \qquad a(\eta) = -\frac{1}{H\eta}, \tag{6}$$

which covers the *expanding* patch for $\eta \in (-\infty, 0)$ and the *contracting* patch for $\eta \in (0, +\infty)$. These patches correspond to opposite orientations of cosmic time and each provides an asymptotic region in which positive-frequency behavior may be unambiguously defined.

A free scalar field decomposes as

$$\phi(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\left[a_{\mathbf{k}}\, u_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}}^{\dagger}\, u_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{7}$$

where the modes satisfy

$$u_k'' + \left(k^2 + a^2m^2 - \frac{a''}{a}\right)u_k = 0. \tag{8}$$

The Bunch–Davies vacuum is defined by requiring that, in the asymptotic past of *either* Poincaré patch, the modes reduce to Minkowski positive-frequency solutions. In the expanding patch, where $\eta \to -\infty$, physical wavelengths lie deep inside the horizon and curvature effects are negligible, so one imposes

$$u_k(\eta) \longrightarrow \frac{1}{\sqrt{2k}}\, e^{-ik\eta}, \qquad (\eta \to -\infty). \tag{9}$$

In the contracting patch, the corresponding asymptotic region is $\eta \to +\infty$. Analytic continuation shows that imposing the same positive-frequency behavior there yields precisely the analytic continuation of the expanding-patch solution. The resulting mode function $u_k(\eta)$ is thus the unique solution that is positive frequency in both asymptotic regimes, propagating smoothly through the de Sitter "neck" connecting the contracting and expanding patches.

This understanding is sharpened by the Euclidean approach. A Wick rotation to the four-sphere S^4 allows one to define the vacuum via the Euclidean path integral on a hemisphere, producing a state that is regular everywhere on the Euclidean manifold. Analytic continuation to Lorentzian signature yields a wavefunctional $\Psi_{\rm BD}$ that naturally covers both Poincaré patches. In the real-time Schwinger–Keldysh contour, the *ket* is evolved forward along the expanding branch while the *bra* evolves backward along the time-reversed contracting branch, ensuring analyticity and de Sitter invariance.

Thus the Bunch–Davies vacuum is the unique state whose mode functions are analytic in the lower half of the complex conformal-time plane and propagate consistently across both contracting and expanding Poincaré patches. It provides the ultraviolet-complete "no-particle" state underlying inflationary cosmology and serves as the standard initial condition for perturbative quantum field theory in de Sitter spacetime.

3 Studied Works

3.1 The In-Out Formalism for In-In Correlators by Donath and Pajer

This paper shows that the correlator's(product of local operators) expectation values using both in-out formalism and in-in formalism are the same, which we can intuitively expect as discussed in the introduction. They prove it for both flat and de Sitter spacetime.

Then the paper explains the mathematical procedure to apply in-out Feynman rules and other calculations commonly done in scattering and collision experiments in the cases of cosmological correlators.

3.1.1 Feynman rules for cosmological correlators in in-out formalism

In the formulation of Donath and Pajer, late-time cosmological correlators in de Sitter space can be computed using an ordinary in-out amplitude built from Feynman rules adapted to the Bunch–Davies vacuum. The analytic continuation of the expanding patch into a contracting patch collapses the Schwinger–Keldysh contour into a single in-out contour, so the diagrammatic rules take a standard Feynman form.

Here external operators correspond to field insertions at late time. Each external leg contributes the late-time limit of the Bunch–Davies mode function,

$$K_k(\eta) \equiv \lim_{\eta \to 0} u_k(\eta), \tag{10}$$

where for a scalar field of mass m,

$$u_k(\eta) = \frac{\sqrt{\pi}}{2} e^{i(\nu+\frac{1}{2})\frac{\pi}{2}} (-\eta)^{3/2} H_\nu^{(1)}(-k\eta), \qquad \nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}. \tag{11}$$

Interaction vertices arise from the bulk interaction action

$$S_{\rm int} = \int d\eta\, d^3x\; a^4(\eta)\, \mathcal{L}_{\rm int}(\phi), \qquad a(\eta) = -\frac{1}{H\eta}. \tag{12}$$

There is *no doubling of vertices* into "±" types, unlike in the Schwinger–Keldysh formalism. For example, a cubic interaction $\lambda\phi^3$ yields the vertex

$$V = -i\lambda \int_{-\infty(1-i\epsilon)}^{+\infty(1+i\epsilon)} d\eta\; a^4(\eta). \tag{13}$$

All internal lines use the single Bunch–Davies Feynman propagator

$$G_F(x, x') = \langle 0_{\rm BD} | T\{\phi(x)\phi(x')\} | 0_{\rm BD} \rangle. \tag{14}$$

In momentum space this takes the form

$$G_F(k; \eta, \eta') = \theta(\eta - \eta')\, u_k(\eta) u_k^*(\eta') + \theta(\eta' - \eta)\, u_k^*(\eta) u_k(\eta'). \tag{15}$$

No Wightman, anti-time-ordered, or mixed propagators appear.

The analytic continuation of the expanding patch into a contracting one collapses the Schwinger–Keldysh contour to a single in-out contour,

$$\eta : -\infty(1 - i\epsilon) \;\longrightarrow\; +\infty(1 + i\epsilon), \tag{16}$$

so all diagrams are evaluated using ordinary Feynman time ordering.

The complete in-out diagrammatic amplitude is

$$\mathcal{A}_{\text{in–out}} = \int \Bigg[\prod_{\text{vertices } i} d\eta_i\, a^4(\eta_i) \Bigg] \Bigg[\prod_{\text{internal lines}} G_F \Bigg] \Bigg[\prod_{\text{external legs } j} K_{k_j} \Bigg]. \tag{17}$$

3.2 New Exceptional EFTs in de Sitter Space from Generalised Energy Conservation by Du and Stefanyszyn

This paper shows that a general propagation amplitude may have the following form.

$$A = A^{(\pm)}_{(k_T \neq 0)} \;+\; A^{(m)}\, \partial^m_{k_T}\, \delta(k_T)\, .$$

Here the first term is eternally energy creating/annihilating and the other term is an energy conserving delta/derivative of delta. But in order to ensure stability, the first term must be set to 0 which is the idea of generalized energy conservation as postulated in this work.

Now applying the in-out formalism we can calculate the amplitude for different fields and using the GEC constraint limit the space of all possible field theories to a great extent. This shows the presence of an "exceptional series" of new scalar field theories.

References

[1] Y. Donath and E. Pajer, *The In–Out Formalism for In–In Correlators*, arXiv:2402.05999.

[2] Z.-Z. Du and D. Stefanyszyn, *New Exceptional EFTs in de Sitter Space from Generalised Energy Conservation*, arXiv:2506.21759.

[3] T. S. Bunch and P. C. W. Davies, *Quantum Field Theory in de Sitter Space: Renormalization by Point Splitting*, Proc. Roy. Soc. Lond. A **360**, 117 (1978).

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