ICTP PWF

Internship Report

Newtonian Structure Formation

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Declaration

I, **Subaita Rahman Supti**, hereby declare that:

1. The work reported in this document is my own original research, carried out under the guidance of my supervisor, **Dr. Rafid Mahbub**, at the ICTP PWF.
2. All sources of information and assistance received during this research have been duly acknowledged.
3. This report has not been submitted, either in whole or in part, for any other degree or diploma at any institution.
4. The content presented herein is authentic and represents my genuine understanding and contribution to the field.

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Acknowledgement

I would like to express my sincere gratitude to all those who contributed to the successful completion of this research internship.

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Subaita Rahman Supti
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Mathematical Notations, Symbols and Formalism

Mathematical Operators

Symbol	Description
∇	Gradient operator
∇^2	Laplacian operator
∂_t	Partial derivative with respect to time
$\dot{X}$	Time derivative: $\dot{X} = \frac{dX}{dt}$
$\ddot{X}$	Second time derivative: $\ddot{X} = \frac{d^2X}{dt^2}$
$\langle \cdot \rangle$	Ensemble or spatial average

Key Equations Reference

Friedmann Equation (Newtonian form):

$$H^2 = \frac{8\pi G}{3}\rho - \frac{\kappa c^2}{R_0^2 a^2}$$

Friedmann Equation (with components):

$$H^2 = H_0^2 \left[\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2} \right]$$

Energy density scaling:

$$\varepsilon_i(a) = \varepsilon_{i,0} a^{-3(1+w_i)}$$

Redshift definition:

$$z = \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{em}}} = \frac{1}{a} - 1$$

Virial theorem:

$$\bar{T} = -\frac{1}{2}\bar{V}$$

Perturbation growth equation (matter era):

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m \delta_m = 0$$

Inflaton equation of motion:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0$$

Fourier Transform and Inverse:

$$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i k x}\, dx,$$
$$f(x) = \int_{-\infty}^{\infty} F(k)\, e^{2\pi i k x}\, dk.$$

Abstract

Abstract

This work explores the physics of Newtonian structure formation in the early universe. It begins with the study of the expanding universe through the Friedmann equation from a Newtonian perspective to model the universe's expansion. The evidence of dark matter was examined via the virial theorem and gravitational lensing. In addition, it discusses the Cosmic Microwave Background and the Inflation field to resolve the naturalness problem. Finally, it concludes how the structure in the universe grows in such a non-uniform pattern.

Keywords: virial theorem, Newtonian mechanics, dark matter, inflation, perturbation, structure formation, cosmology.

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1 Introduction

Observations of the early universe reveal that small density inhomogeneities were already present shortly after the Big Bang. Although the cosmic microwave background (CMB) has a nearly uniform temperature of $T \simeq 2.725\,\mathrm{K}$, it exhibits tiny anisotropies at the level of $\Delta T/T \sim 10^{-5}$. These minute temperature variations trace the initial density perturbations that later evolved, under gravitational instability, into the large-scale cosmic structures observed today.

However, on sufficiently large scales ($\sim 100\,\mathrm{Mpc}$), the universe appears *homogeneous* and *isotropic*. *Isotropy* means that the universe looks the same in every direction, while *homogeneity* means that the average properties of matter and radiation are the same at every location. These correspond to rotational and translational symmetries, respectively, and form the foundation of the Cosmological Principle.

In this work, we focus on the formation of structures arising from these primordial density perturbations. Using observations of CMB temperature anisotropies and large-scale galaxy distributions, we examine how initial fluctuations evolved after matter domination and how overdense regions expanded more slowly than the universe as a whole. This process ultimately led to the non-uniform structures seen in the present-day universe.

Timeline of the Universe

The early universe underwent a series of significant periods, including phase transitions, that shaped its present structure and composition. We can describe the history in terms of the physical events and processes that occurred during each epoch. A brief description of these is provided below:

- **Planck era** ($t < 10^{-43}$ s): During this epoch, quantum gravitational effects dominated the dynamics of the universe. The classical description of spacetime breaks down, and a theory of quantum gravity is required to describe physical processes.
- **Grand Unification Epoch** (10^{-43}–10^{-36} s): In this period, the strong, weak, and electromagnetic forces may have been unified into a single fundamental interaction. The universe was extremely hot and dense, with rapid energy fluctuations.
- **Inflationary Epoch** ($t \sim 10^{-36}$ s): The universe underwent an exponential expansion, increasing its size by many orders of magnitude in a tiny fraction of a second. This process smoothed out inhomogeneities, drove the universe toward flatness, and set the initial conditions for cosmic structure formation.

- **Electroweak Phase Transition** ($t \sim 10^{-12}$ s): The electromagnetic and weak nuclear forces separated into distinct interactions. Massive elementary particles acquired mass through the Higgs mechanism during this transition.
- **Quark and Hadron Epochs** (10^{-12}–10^{-6} s): As the universe continued to cool, quarks combined to form hadrons such as protons and neutrons. This marked the transition from a quark-gluon plasma to a hadron-dominated universe.
- **Lepton Epoch** ($t \sim 1$ s): Leptons, such as electrons and neutrinos, dominated the energy density of the universe. Most leptons and anti-leptons annihilated each other, leaving a small excess of electrons that would later combine with protons to form atoms.
- **Big Bang Nucleosynthesis** ($t \sim 3$ min): Light nuclei such as hydrogen, helium, and small amounts of lithium formed through nuclear fusion reactions.
- **Photon Epoch** ($t \sim 10^4$ yr): The universe became radiation-dominated, with photons frequently interacting with charged particles. The matter and radiation were tightly coupled during this period.
- **Recombination** ($t \sim 3.8 \times 10^5$ yr): Electrons combined with protons to form neutral hydrogen atoms. As a result, photons decoupled from matter and began to travel freely through space, forming the Cosmic Microwave Background (CMB) radiation.
- **Dark Ages** (10^5–10^8 years): After recombination, the universe became dark and neutral, with no stars or galaxies. This period persisted until the first luminous objects began to form.
- **Reionization and Structure Formation** (10^8–10^9 yr): The first stars and galaxies formed, emitting high-energy radiation that reionized the intergalactic medium. Cosmic structures such as galaxies and clusters of galaxies began to develop during this time.
- **Present Day** ($t \approx 13.8$ Gyr): The universe continues to expand, with the expansion rate accelerating because of dark energy. Large-scale structures such as galaxies, clusters, and cosmic voids characterize the current universe.

2 The Expanding Universe

In the 1920, Edwin Hubble's observation revealed that distant galaxies are receding from us, with velocities proportional to distances, which is now known as Hubble's law. This discovery provided the first clear evidence that the universe is expanding. The rate and nature of the universe are governed by the balance between the energy content of the universe and its overall geometry. This relationship is mathematically expressed through the Friedmann equations, which describe the time evolution of the cosmic scale factor. Although these equations originate from Einstein's field equations of general relativity, their essential form can be derived using Newtonian mechanics under the assumptions of homogeneity and isotropy. In this section, we present the Newtonian formulation of the Friedmann equations and discuss their implications through numerical results that trace the dynamical evolution of the universe.

Suppose that we consider a particular absorption or emission line whose wavelength, as measured in a laboratory here on Earth, is $\lambda_{\rm em}$. The wavelength we measure for the same line in the observed spectrum of a distant galaxy, $\lambda_{\rm ob}$, will not, in general, be the same. We say that the galaxy has a redshift z, given by

$$z \equiv \frac{\lambda_{\rm ob} - \lambda_{\rm em}}{\lambda_{\rm em}} \tag{2.1}$$

When $z < 0$, this quantity is blueshifted rather than redshifted. Moreover, Hubble noted that the more distant galaxies had higher redshifts, pointing towards the fact that the farther the galaxies are, the greater their speeds of recession.

The simplest form of Hubble's law can be expressed as

$$v = H_0 D \tag{2.2}$$

where v is the recession speed of the object and D is the proper distance between the object and the observer.

as illustrated in Figure 1.

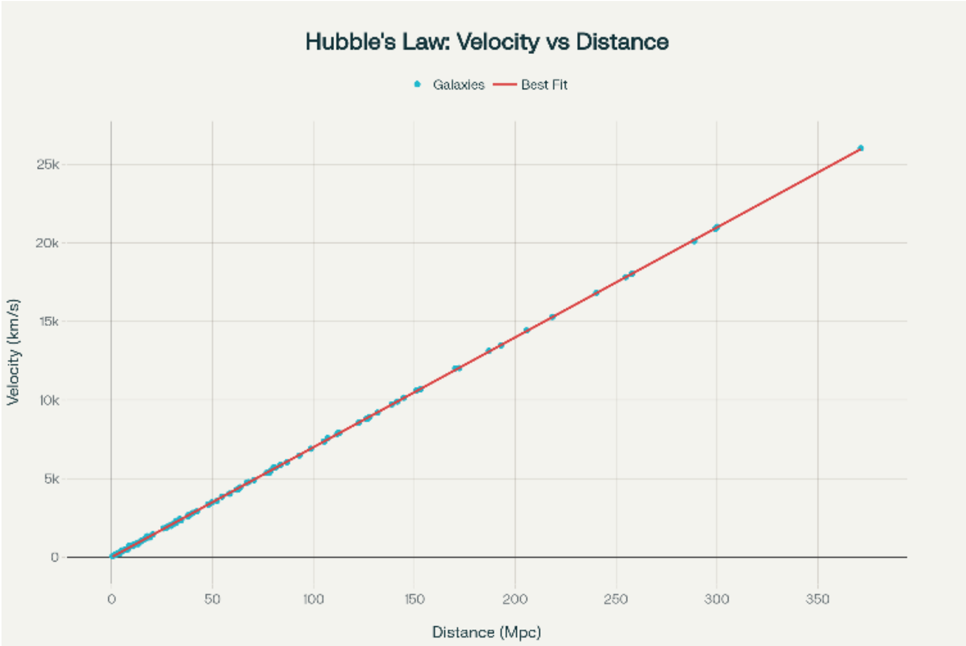

Figure 1: Hubble's Law: Relation between velocity and distance of galaxies.

2.1 The Friedmann Equation

The expansion or contraction of the universe is first demonstrated from a Newtonian perspective, starting with Newton's laws of gravity and motion, as Newtonian structure formation is studied. A homogeneous sphere of matter with total mass M_s, which is constant in time, is considered. The sphere is assumed to expand or contract isotropically, with a radius $R_s(t)$. A test mass is placed on the surface of the sphere. The force acting on the test mass is then determined by the mass inside the radius. From Newton's law of gravity, the force on the test mass is

$$F = -\frac{GM_s m}{R_s^2}. \tag{2.3}$$

Using Newton's second law, the radial acceleration of the surface is

$$\frac{d^2 R_s}{dt^2} = -\frac{GM_s}{R_s^2}. \tag{2.4}$$

Multiply both sides by $\dfrac{dR_s}{dt}$ and integrate:

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2\right] = \frac{d}{dt}\left(\frac{GM_s}{R_s}\right), \tag{2.5}$$

so that integrating once gives the energy conservation per unit mass;

$$\frac{1}{2}\left(\frac{dR_s}{dt}\right)^2 = \frac{GM_s}{R_s} + E, \tag{2.6}$$

where E is a constant (the total mechanical energy). We may interpret $\varepsilon_{\text{kin}} = \frac{1}{2}\dot{R}_s^2$ and $\varepsilon_{\text{pot}} = -GM_s/R_s$ as the kinetic and potential energies per unit mass, respectively.

Because the sphere's mass is constant, we can write

$$M_s = \frac{4\pi}{3}\rho(t)\, R_s^3. \tag{2.7}$$

For an isotropic expansion, we define the scale factor $a(t)$ and the comoving radius r_s by

$$R_s(t) = a(t)\, r_s. \tag{2.8}$$

Then $dR_s/dt = r_s\dot{a}$. Substituting into the energy equation and using the expression for M_s yields

$$\frac{1}{2}r_s^2\dot{a}^2 = \frac{4\pi G}{3}\rho(t)\, a^2 r_s^2 + E. \tag{2.9}$$

Divide both sides by $\frac{1}{2}r_s^2 a^2$ to obtain

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{r_s^2}\frac{1}{a(t)^2}. \tag{2.10}$$

Equation (2.10) is the Newtonian form of the Friedmann equation. The last term acts like a curvature term scaling as a^{-2} where its sign depends on the sign of the specific energy E:

- $E > 0$: the RHS stays positive → expansion never stops (open-like).
- $E = 0$: marginal/boundary case (expansion asymptotes to zero velocity at infinite time).
- $E < 0$: expansion stops at some a_{max} and contraction follows (closed-like).

If we identify

$$\frac{2E}{r_s^2} \equiv -\frac{\kappa}{R_0^2}, \tag{2.11}$$

then the Newtonian expression matches the standard relativistic Friedmann equation (with energy density $\varepsilon = \rho$):

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\varepsilon(t) - \frac{\kappa}{R_0^2}\frac{1}{a^2}. \tag{2.12}$$

Here $\kappa = +1, 0, -1$ denotes positive, zero, or negative spatial curvature, respectively, and R_0 is a radius curvature scale. The Newtonian cases map to relativistic curvature as:

- $E < 0 \longleftrightarrow \kappa = +1$ (positive curvature),
- $E = 0 \longleftrightarrow \kappa = 0$ (flat),
- $E > 0 \longleftrightarrow \kappa = -1$ (negative curvature).

The Hubble parameter is defined by $H \equiv \dot{a}/a$. Rewriting (2.12) as

$$H(t)^2 = \frac{8\pi G}{3}\varepsilon(t) - \frac{\kappa}{R_0^2 a^2}, \tag{2.13}$$

We can evaluate it today ($t = t_0$) to relate the present Hubble constant H_0, the present energy density ε_0, and the curvature:

$$H_0^2 = \frac{8\pi G}{3}\varepsilon_0 - \frac{\kappa}{R_0^2}. \tag{2.14}$$

For a spatially flat universe ($\kappa = 0$) this reduces to

$$H(t)^2 = \frac{8\pi G}{3}\varepsilon(t). \tag{2.15}$$

From this, we define the critical energy density as

$$\varepsilon_c(t) \equiv \frac{3}{8\pi G}H^2. \tag{2.16}$$

Comparing the actual energy density ε to the critical density ε_c, we can determine the geometry of the universe:

$$\kappa = \begin{cases} +1, & \text{if } \varepsilon > \varepsilon_c \quad \text{(closed universe)} \\ 0, & \text{if } \varepsilon = \varepsilon_c \quad \text{(flat universe)} \\ -1, & \text{if } \varepsilon < \varepsilon_c \quad \text{(open universe)} \end{cases}$$

The present-day value of the Hubble parameter highly depends on which set of observational data from which it has been derived. Two such values are –

$$H_0 \simeq \begin{cases} 68\ \mathrm{km\,s^{-1}\,Mpc^{-1}}, \\ 73\ \mathrm{km\,s^{-1}\,Mpc^{-1}}\,. \end{cases} \tag{2.17}$$

The first one is obtained through Type Ia supernovae data. There is a 5σ discrepancy between the two values. This problem is known as the 'Hubble tension'(Hu, 2023) .

When written in terms of the density parameter $\Omega(t) \equiv \varepsilon(t)/\varepsilon_c(t)$, the Friedmann equation becomes

$$\Omega(t) - 1 = \frac{\kappa}{R_0^2 H^2 a^2}. \tag{2.18}$$

A useful property of this idealized model is that if $\Omega = 1$ at any moment, it is maintained at 1 for all times. Similarly, the sign of $\Omega - 1$ is conserved over time for simple fluid contents. Present combined observational constraints indicate that Ω_0 is found to be very close to unity, within about a percent. In the present universe, matter is strongly dominant over radiation. In a universe with many components, the Friedmann equation can be written in the form

$$\dot{a}^2 = \frac{8\pi G}{3} \sum_i \varepsilon_{i,0}\, a^{-1-3w_i} - \frac{\kappa}{R_0^2}\,. \tag{2.19}$$

2.2 Cosmic redshift and the lookback time

We know that our universe contains matter and radiation components for which the energy densities scale as $\varepsilon_m = \varepsilon_{0,m} a^{-3}$ and $\varepsilon = \varepsilon_{0,r} a^{-4}$ respectively. Furthermore, there is a dark energy contribution to the total energy budget of the universe. Current evidence indicates that its energy density is constant $\varepsilon_\Lambda = \varepsilon_{\Lambda,0} = \text{constant}$.

We will consider a universe with contributions from matter, radiation, and the cosmological constant. In our universe, we expect the Friedmann equation to take the form

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2}\,, \tag{2.20}$$

where

$$\Omega_{r,0} = \frac{\varepsilon_{r,0}}{\varepsilon_{c,0}}, \quad \Omega_{m,0} = \frac{\varepsilon_{m,0}}{\varepsilon_{c,0}}, \quad \Omega_{\Lambda,0} = \frac{\varepsilon_{\Lambda,0}}{\varepsilon_{c,0}}, \quad \text{and} \quad \Omega_0 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0}\,.$$

Multiplying both sides of the equation by a^2 yields

$$H_0^{-1}\dot{a} = \left[\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)\right]^{1/2} . \tag{2.21}$$

The cosmic time t as a function of the scale factor a can be found by integrating the Eq. (2.22) –

$$H_0 t = \int_0^a \frac{da}{\left[\Omega_{r,0}a^{-2} + \Omega_{m,0}a^{-1} + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)\right]^{1/2}} . \tag{2.22}$$

Equivalently, in terms of the redshift z, where $1+z = a^{-1}$, we can write

$$H_0 t = \int_z^\infty \frac{dz}{\left[\Omega_{r,0}(1+z)^3 + \Omega_{m,0}(1+z)^2 + \Omega_{\Lambda,0} + (1-\Omega_0)\right]^{1/2}} . \tag{2.23}$$

The resulting lookback time as a function of redshift is shown in Figure 2.

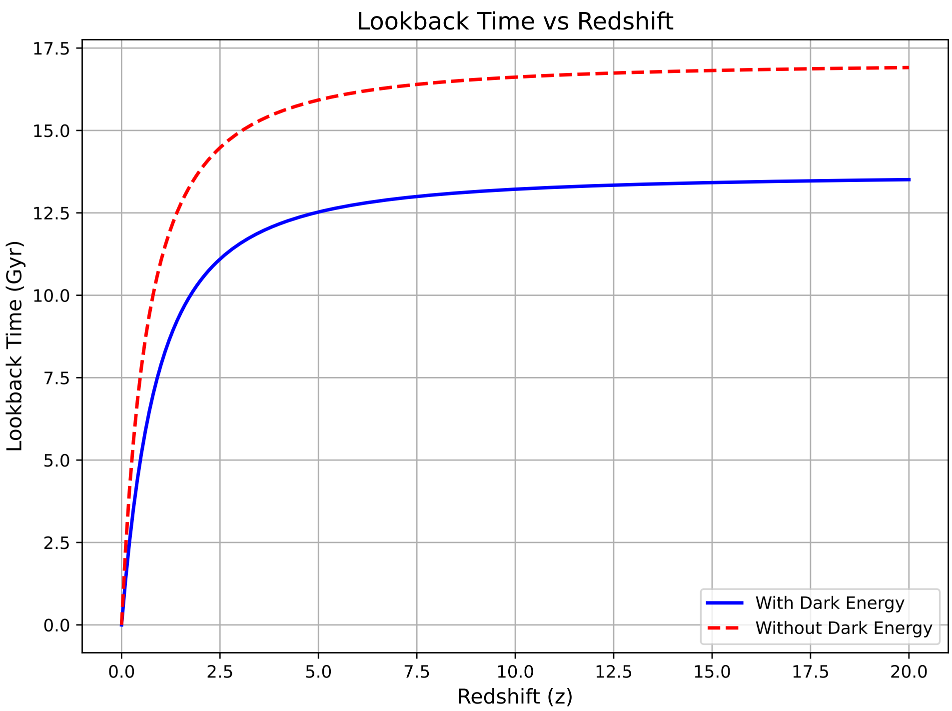

Figure 2: Lookback time as a function of redshift. The blue curve represents the model with dark energy, while the red dashed curve represents the model without dark energy.

2.3 Evolution of Energy Density

The Friedmann equation represents the statement of energy conservation. For fluids of cosmological importance, the equation of state can be written as

$$P = w\varepsilon$$

where w is the *equation of the state parameter*. The equation of state parameter w characterizes different components of cosmic fluid through their pressure-to-energy-density ratio:

$$w = \frac{P}{\varepsilon}$$

In the Newtonian approximation, the Friedmann equation can be interpreted as conservation of energy:

$$\frac{1}{2}\dot{a}^2 - \frac{8\pi G\rho}{3}a^2 = \text{constant}$$

This represents **kinetic energy minus potential energy** for a test particle at the edge of an expanding sphere. The kinetic energy term $\frac{1}{2}\dot{a}^2$ represents the kinetic energy per unit mass of expansion, and the potential energy term $-\frac{8\pi G\rho}{3}a^2$ represents the (negative) gravitational potential energy per unit mass.

For non-relativistic matter (dust), $w = 0$, which means $P = 0$ (pressureless), $\rho \propto a^{-3}$, and energy is dominated by rest mass. For radiation (relativistic particles), $w = \frac{1}{3}$, so $P = \frac{\varepsilon}{3}$, $\rho \propto a^{-4}$, and energy is equally distributed between kinetic energy and momentum flux. For dark energy (cosmological constant), $w = -1$, giving $P = -\varepsilon$, $\rho = \text{constant}$, negative pressure that causes accelerated expansion, and energy density remains constant as the universe expands.

For an adiabatic expansion (entropy conserved), the first law of thermodynamics gives:

$$\frac{d\varepsilon}{dt} = -3H(\varepsilon + P)$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. This shows how both energy density **and pressure** contribute to the deceleration of expansion. Using $P = w\varepsilon$, this becomes:

$$\frac{d\varepsilon}{dt} = -3H\varepsilon(1 + w)$$

which integrates to:

$$\varepsilon \propto a^{-3(1+w)}$$

This demonstrates that w fundamentally determines how energy density evolves with cosmic expansion, encoding the balance between kinetic (expansion) and potential (gravitational) energies in the cosmological fluid.

The equation of state of photons or any other relativistic gas is

$$P_{\text{rel}} = \frac{1}{3}\varepsilon_{\text{rel}}\,. \tag{2.24}$$

We will refer to the component of the universe that consists of photons and other relativistic particles as *radiation*. The case $w < -\frac{1}{3}$ is of interest, because a component (cosmological constant) with $w < -\frac{1}{3}$ provides a positive acceleration.

The universe contains nonrelativistic matter and radiation. Thus, the universe contains components with both $w = 0$ and $w = \frac{1}{3}$, and the cosmological constant has $w = -1$. We may then write the total energy density ε as the sum of the energy density of the different components:

$$\varepsilon = \sum_i \varepsilon_i\,. \tag{2.25}$$

The total pressure P is the sum of the pressures of the different components:

$$P = \sum_i w_i \varepsilon_i\,. \tag{2.26}$$

The fluid equation must hold for each component separately. The component with equation-of-state parameter w_i obeys the equation

$$\dot{\varepsilon}_i + 3\frac{\dot{a}}{a}(\varepsilon_i + P_i) = 0\,, \tag{2.27}$$

or equivalently,

$$\dot{\varepsilon}_i + 3\frac{\dot{a}}{a}(1 + w_i)\varepsilon_i = 0. \tag{2.28}$$

If we assume that w_i is constant, then the solution to Eq. (2.28) is given by

$$\varepsilon_i(a) = \varepsilon_{i,0}\, a^{-3(1+w_i)}. \tag{2.29}$$

The energy density ε_m associated with nonrelativistic matter decreases as the universe expands with this dependence. The energy density in radiation, ε_r, drops at a steeper rate.

The reasons behind these scalings are as follows:

- **Matter:** Energy scales as a^{-3} since expansion of the universe is proportional to a^3. Since matter is nonrelativistic, its energy is dominated by rest mass, which is conserved. Therefore, the number density of matter particles simply dilutes as the expansion of the universe.
- **Radiation:** Energy scales as a^{-4} due to the expansion of the universe and the redshift of wavelengths. Radiation experiences dilution from the expansion, contributing a factor of a^{-3}. In addition, relativistic particles undergo cosmological redshift, causing their wavelengths to stretch proportionally to the scale factor a, which leads to their individual energies decreasing by an additional factor of a^{-1}.
- **Cosmological constant:** Energy scales as a^0, which tells us that for a cosmological constant (dark energy), the energy density stays constant.

Table 1 summarizes the variation of energy density for different components of the universe and their corresponding equations of state.

Table 1: Variation of energy density ε for different equations of state (EoS).

Component	**Equation of State (EoS)**	**EoS Parameter w**	**Scaling Law for ε**
Non-relativistic matter ("dust")	$p = 0$	$w = 0$	$\varepsilon \propto a^{-3}$
Radiation	$p = \frac{1}{3}\rho$	$w = \frac{1}{3}$	$\varepsilon \propto a^{-4}$
Cosmological constant	$p = -\rho$	$w = -1$	$\varepsilon =$ constant
Stiff matter	$p = \rho$	$w = 1$	$\varepsilon \propto a^{-6}$
Quintessence	$p = w\rho,\ -1 < w < -\frac{1}{3}$	Variable	$\varepsilon \propto a^{-3(1+w)}$
General perfect fluid	$p = w\rho$	Constant w	$\varepsilon \propto a^{-3(1+w)}$

Current observational data suggest a universe in which the density parameter for matter is currently $\Omega_{m,0} \approx 0.31$, while that for the cosmological constant is $\Omega_{\Lambda,0} \approx 0.69$. Furthermore, we have $\Omega_{r,0} = 9 \times 10^{-5}$ radiation $\Omega_{m,0} = 0.31$ in non-relativistic matter, and

$$\Omega_{\Lambda,0} = 1 - \Omega_{r,0} - \Omega_{m,0} \approx 0.69\,. \tag{2.30}$$

This model is known as the "Λ-CDM" or "concordance" model.

At present, the ratio of the energy density in Λ to the energy density in matter is

$$\frac{\varepsilon_{\Lambda,0}}{\varepsilon_{m,0}} = \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} \approx \frac{0.69}{0.31} \approx 2.23\,. \tag{2.31}$$

In the cosmologist's language, the cosmological constant is "dominant" over matter today. In the past, when the scale factor was smaller, the ratio of densities was

$$\frac{\varepsilon_\Lambda(a)}{\varepsilon_m(a)} = \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} a^3 \,. \tag{2.32}$$

The moment of matter–Λ equality occurred when the scale factor was

$$a_{m\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \approx \left(\frac{0.31}{0.69}\right)^{1/3} \approx 0.766 \,. \tag{2.33}$$

Similarly, the ratio of the energy density in matter to the energy density in radiation is currently

$$\frac{\varepsilon_{m,0}}{\varepsilon_{r,0}} = \frac{\Omega_{m,0}}{\Omega_{r,0}} \approx \frac{0.31}{9 \times 10^{-5}} \approx 3400 \,. \tag{2.34}$$

3 The Cosmic Microwave Background

The cosmic microwave background (CMB) is the first piece of evidence for the hot Big Bang—the idea that the early universe was filled with a hot, dense plasma composed of all the Standard Model particles. As shown in Figure 3, the CMB exhibits tiny temperature fluctuations across the sky, which trace the primordial density variations that eventually gave rise to cosmic structures. The mean energy per CMB photon is tiny compared to the energy required to break up an atomic nucleus. Still, the mean photon energy is comparable to the rotational energy of a small molecule such as H_2O. Thus, CMB photons can zip along for more than 13 billion years through tenuous intergalactic gas, then be absorbed a microsecond away from the Earth's surface.

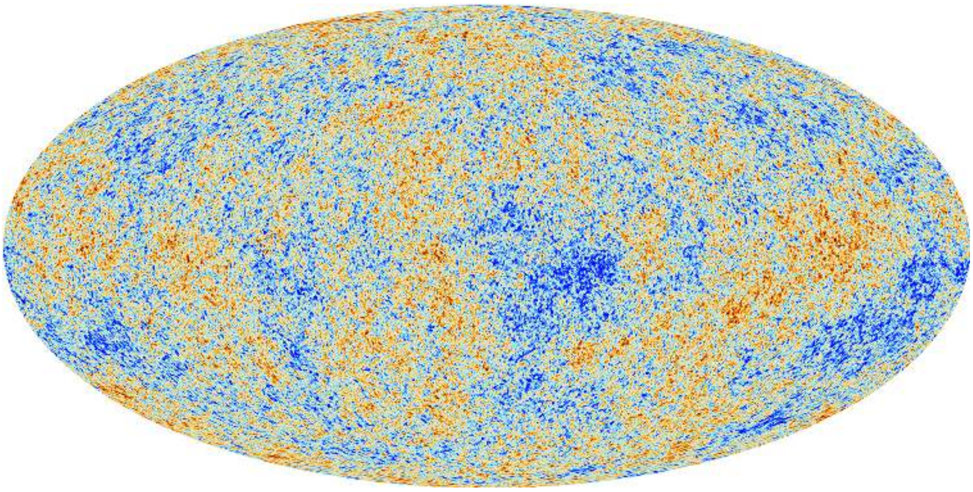

Figure 3: Cosmic Microwave Background (CMB) map from the Planck satellite. This map shows the temperature fluctuations across the sky, which correspond to the primordial density fluctuations that grew into cosmic structures.

The CMB spectrum was first measured over a wide range of wavelengths by the Cosmic Background Explorer (COBE) satellite. Subsequently, more detailed observations were made by the WMAP and Planck satellites. At any angular position (θ, ϕ) on the sky, the spectrum of the cosmic microwave background is very close to that of an ideal blackbody. The CMB also shows a dipole distortion in temperature: on one half of the sky, the spectrum is slightly redshifted to lower temperatures, while on the other half, it is slightly blue-shifted to higher temperatures. This Doppler shift is caused by the net motion of the observer (such as the WMAP spacecraft) relative to the frame of reference in which the CMB is isotropic.

After subtracting the dipole distortion of the CMB, the remaining temperature fluctuations are small in amplitude. Let's assume the temperature of the CMB at a given point on the sky is $T(\theta, \phi)$. The mean temperature, averaging over all locations, is

$$\langle T \rangle = \frac{1}{4\pi} \int T(\theta, \phi) \sin\theta \, d\theta \, d\phi \tag{3.1}$$

$$= 2.7255 \text{ K} \tag{3.2}$$

The dimensionless temperature fluctuation at a given point (θ, ϕ) on the sky is

$$\frac{\delta T}{T}(\theta, \phi) \equiv \frac{T(\theta, \phi) - \langle T \rangle}{\langle T \rangle} \tag{3.3}$$

After subtracting the Doppler dipole, the root mean square temperature fluctuation found by COBE was

$$\left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-5} \tag{3.4}$$

The distortions on a smaller angular scale tell us that the universe was not perfectly homogeneous at the time of last scattering ($z \approx 1090$). The angular size of the temperature fluctuations reflects, in part, the physical size of the density and velocity fluctuations at $z \approx 1090$.

The angular size $\delta\theta$ of a temperature fluctuation in the CMB is related to a physical size l on the last scattering surface by the relation

$$d_A = \frac{l}{\delta\theta} \tag{3.5}$$

where d_A is the angular diameter distance to the last scattering surface. Since the last scattering surface is at a redshift $z_{\text{ls}} = 1090 \gg 1$, we can write

$$d_A \approx \frac{d_{\text{hor}}(t_0)}{z_{\text{ls}}} \tag{3.6}$$

The horizon distance is an interesting scale which can be connected with the time of last scattering by,

$$d_{\text{hor}}(t_{\text{ls}}) = a(t_{\text{ls}})\, c \int_0^{t_{\text{ls}}} \frac{\mathrm{d}t}{a(t)} \tag{3.7}$$

Since the last scattering took place long before the cosmological constant plays a significant role in the expansion, the scale factor is appropriate for a universe containing just radiation and matter. Therefore, we get the horizon distance of

$$d_{\text{hor}}(t_{\text{ls}}) = 2.24\, ct_{\text{ls}} = 0.251\,\text{Mpc} \tag{3.8}$$

A patch of the last scattering surface with this physical size will have an angular size given by

$$\theta_{\rm hor} = \frac{d_{\rm hor}(t_{\rm ls})}{d_A} = \frac{0.251\,{\rm Mpc}}{12.8\,{\rm Mpc}} \approx 0.020\,{\rm rad} \approx 1.1^\circ \tag{3.9}$$

The fluctuations on large angular scales, with $\theta > \theta_{\rm hor}$, are generated from physical processes that are different that those for $\theta < \theta_{\rm hor}$. The large-scale fluctuations with $\theta > \theta_{\rm hor}$ arise from the gravitational effect of density fluctuations in the distribution of nonbaryonic dark matter. The density of nonbaryonic dark matter at the time of the last scattering, since $\varepsilon_{\rm dm} \propto a^{-3} \propto (1+z)^3$, was

$$\varepsilon_{dm}(z_{ls}) = \Omega_{dm,0}\varepsilon_{c,0}(1+z_{ls})^3 \approx 1.7 \times 10^{12}\ {\rm MeV.m^{-3}} \tag{3.10}$$

where the density of the baryonic matter is

$$\varepsilon_{bary}(z_{ls}) = \Omega_{bary,0}\varepsilon_{c,0}(1+z_{ls})^3 \approx 3.1 \times 10^{11}\ {\rm MeV.m^{-3}} \tag{3.11}$$

Finally, the density of photons at the time of last scattering was

$$\varepsilon_{\gamma}(z_{ls}) = \Omega_{\gamma,0}\varepsilon_{c,0}(1+z_{ls})^4 \approx 3.9 \times 10^{11}\ {\rm MeV.m^{-3}} \tag{3.12}$$

Thus, we can conclude that at the time of last scattering, $\varepsilon_{\rm dm} > \varepsilon_\gamma > \varepsilon_{\rm bary}$, Non-baryonic matter dominated the energy density ε, and hence the gravitational potential, of the universe at the time of last scattering.

The energy density of the dark matter at the time of last scattering can be expressed as

$$\varepsilon(t,\bar{r}) = \bar{\varepsilon}(t) + \delta\varepsilon(t,\bar{r}) \tag{3.13}$$

Here, $\bar{\varepsilon}$ is the average energy density of the dark matter, and $\delta\varepsilon$ is the local deviation from the mean.

In the Newtonian approximation, the spatially varying component of the energy density, $\delta\varepsilon$, gives rise to a spatially varying gravitational potential $\delta\Phi$, which is related by Poisson's equation:

$$\nabla^2\delta\Phi = 4\pi G a^2\,\delta\varepsilon \tag{3.14}$$

These fluctuations in the density have given rise to fluctuations in the gravitational potential.

4 Dark Matter

The temperature fluctuations of the cosmic microwave background radiation depend on the baryon-to-photon ratio when the universe was approximately 250,000 years old. In addition, the efficiency of nucleosynthesis depends on the baryon-to-photon ratio. From this information, we find the density parameter of baryonic matter in the present universe to be

$$\Omega_{bary,0} = 0.048 \pm 0.003$$

However, observations indicate a total matter density of $\Omega_{m,0} \approx 0.31$. Thus, baryonic matter makes up merely about 15% of the total matter content of the universe. The remaining 85% cannot be explained by ordinary matter alone. This discrepancy naturally raises a key question: how do we account for the additional matter required by observations, given that it does not exist in baryonic form?

4.1 Dark Matter in Clusters

Fritz Zwicky, when studying the Coma cluster of galaxies, found that the dispersion in the radial velocity of the cluster's galaxies was much larger than what calculations from luminous matter would allow (around 1000 km s^{-1}). To hold the cluster together, the Coma cluster must contain a large amount of dark matter.

The weight of a collection of objects that are far away can be approximated using the virial theorem. Suppose a cluster of galaxies consists of N galaxies, each of which can be approximated as a point mass, with mass m_i ($i = 1, 2, \ldots, N$), position $\vec{x}_i$, and velocity $\dot{\vec{x}}_i$. The motion of a galaxy is given by

$$\ddot{\vec{x}}_i = G \sum_{j \neq i} m_j \frac{\vec{x}_j - \vec{x}_i}{|\vec{x}_j - \vec{x}_i|^3}. \tag{4.1}$$

Here we assume that the cluster is an isolated system and that the gravitational acceleration due to external matter is negligibly small.

The time average of the kinetic energy T and potential energy V can be connected through

$$\bar{T} = -\frac{1}{2}\bar{V}. \tag{4.2}$$

The moment of inertia of the system is denoted by

$$I = \frac{1}{2}\sum_i m\vec{x}_i \cdot \vec{x}_i, \tag{4.3}$$

and its time derivative is

$$\dot{I} = \sum_i \vec{p}_i \cdot \vec{x}_i, \tag{4.4}$$

where $\vec{p}_i$ is the momentum of the i^{th} particle. The quantity $\dot{I}$ is the virial. The correct choice is to pick the origin to be the centre of mass.

The time derivative of the virial is then

$$\begin{aligned}\ddot{I} &= \sum_i \dot{\vec{p}}_i \cdot \vec{x}_i + \sum_i \vec{p}_i \cdot \dot{\vec{x}}_i \\ &= \sum_i \vec{F}_i \cdot \vec{x}_i + 2T. \end{aligned} \tag{4.5}$$

The force $\vec{F}_i$ on the i^{th} particle is determined by the potential V_{ij} via

$$\begin{aligned}\vec{F}_i = -\sum_{j\neq i} \nabla_i V_{ij} \Rightarrow \sum_i \vec{F}_i \cdot \vec{x}_i &= -\sum_{i<j} \nabla_i V_{ij} \cdot \vec{x}_i - \sum_{j<i} \nabla_i V_{ij} \cdot \vec{x}_i \\ &= -\sum_{i<j} \nabla_i V_{ij} \cdot \vec{x}_i - \sum_{i<j} \nabla_j V_{ji} \cdot \vec{x}_j \\ &= -\sum_{i<j} \nabla_i V_{ij} \cdot (\vec{x}_i - \vec{x}_j) \end{aligned} \tag{4.6}$$

We can use the explicit form of the potential to find

$$-\sum_{i<j} \nabla_i V_{ij} \cdot (x_i - x_j) = -\sum_{i<j} V_{ij} = V \tag{4.7}$$

Now we get the time variation of the virial as

$$\ddot{I} = V + 2T \tag{4.8}$$

Now, the time average is defined by

$$\bar{X} = \lim_{t\to\infty} \int_0^t X(t')dt' \tag{4.9}$$

We want to make some assumptions. First, we will assume that there are N galaxies, all of which have the same mass m. Suppose the system is "self-averaging", which means the average over many galaxies is a proxy for averaging over time, so that, for example,

$$\bar{T} \approx \langle T \rangle = \frac{1}{2N} \sum_{i=1}^{N} m v_i^2 \tag{4.10}$$

We don't need to wait several billion years to perform the time average due to this advantage. The virial theorem then reads

$$2 \langle T \rangle = m \left\langle v^2 \right\rangle \approx \langle V \rangle \approx \frac{1}{2} G m^2 N \left\langle \frac{1}{r} \right\rangle \tag{4.11}$$

$\langle 1/r \rangle$ is the average inverse distance between galaxies, and by replacing N-1 with N, we get the expression for the total mass of the galaxy cluster.

$$Nm \approx \frac{2 \langle v^2 \rangle}{G \langle 1/r \rangle} \tag{4.12}$$

We can estimate the mass of each galaxy by counting its stars, which is inferred from its luminosity. This provides two independent methods to determine the mass, which can then be compared. It is found that the virial mass exceeds the luminosity mass by a factor of a few hundred. This discrepancy is attributed to dark matter.

Further evidence for the existence of dark matter comes from gravitational lensing. The mass of a cluster can be determined from the amount of light bending. Most of the matter detected through gravitational lensing is dark, and, like the galaxies, the dark matter components have passed through each other seemingly unaffected by the collision. The interpretation is that dark matter interacts weakly, both with itself and with baryonic matter.

4.2 Vera Rubin's Galactic Rotation Curves

Vera Rubin's observations of galactic rotation curves provided strong evidence for the existence of dark matter. The rotation curves of galaxies, particularly the flatness of the curves at large distances from the galactic center, suggest the presence of an unseen mass, which has gravitational interaction but no electromagnetic interaction.

The rotation velocity $v(r)$ at a distance r from the center of a galaxy can be derived from the balance between the gravitational force and the centripetal force acting on stars. For a point mass at the center of a galaxy, the velocity would decrease as $v(r) \propto r^{-1/2}$. However, Rubin's observations indicate that the rotation velocity remains constant at large r, suggesting that the mass distribution is not concentrated at the center.

The key equation for the rotation curve in a galaxy is:

$$v(r) = \sqrt{\frac{GM(r)}{r}},$$

where: $v(r)$ is the orbital velocity at radius r, $M(r)$ is the mass enclosed within radius r, G is the gravitational constant.

If the mass distribution is dominated by visible matter (such as stars), the velocity should drop off at large r. However, the observed flat rotation curves imply that the mass continues to increase with radius, suggesting the presence of dark matter. This is typically modeled by adding a dark matter halo, which causes the mass $M(r)$ to grow with radius more slowly, maintaining the constant rotation velocity.

$$M(r) \propto r, \tag{4.13}$$

which leads to the observed flat rotation curves (Figure 4), provides clear evidence for the presence of dark matter. The discrepancy between the predicted and observed rotation velocities in the outer regions of galaxies provides direct evidence for the presence of dark matter.

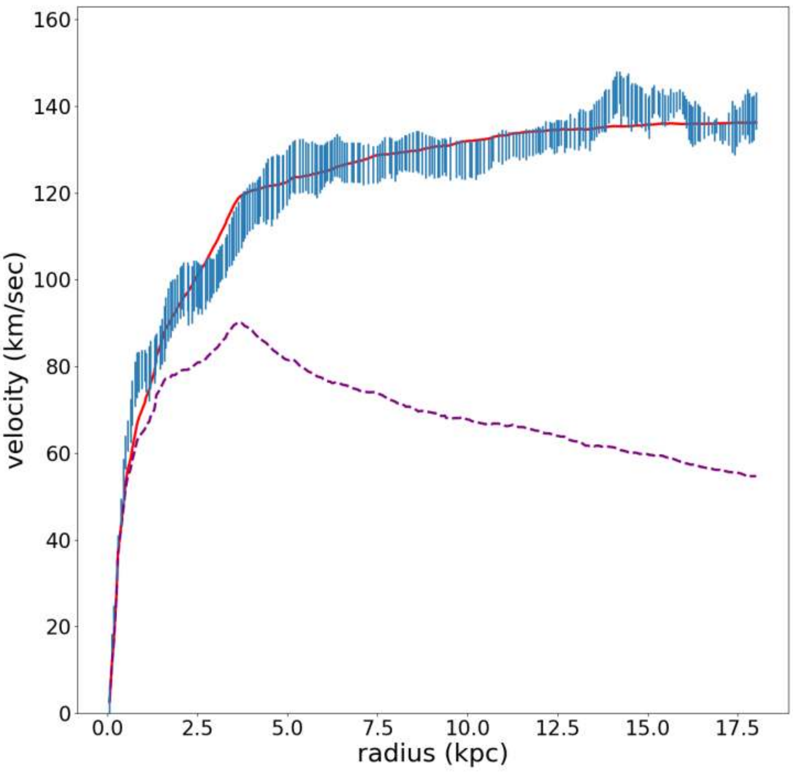

Figure 4: NGC 2403. Blue points with error bars: Rotation curve data. Red line: RCFM fit. Dashed purple line: luminous mass model. Rotation curve data and luminous mass models from de Blok et al. (2008a) for NGC 2403 and Sofue-Xue-Jiao for the Milky Way.

5 Inflation

After the discovery of the cosmic microwave background (CMB), it was realized that the standard Hot Big Bang model has underlying issues known as the *flatness problem*, the *horizon problem*, and the *monopole problem*.

5.1 The Flatness and Horizon Problems

The spatial curvature of the universe is related to the density parameter Ω by the Friedmann equation:

$$1 - \Omega(t) = -\kappa \left(\frac{c/H(t)}{a(t)R_0} \right)^2 \tag{5.1}$$

At the present moment, the density parameter and the curvature are linked by:

$$1 - \Omega_0 = -\kappa \left(\frac{c/H_0}{R_0} \right)^2 \tag{5.2}$$

For most of the past 14 billion years, the universe was matter-dominated. In this case,

$$\frac{\rho_k(t)}{\rho_m(t)} = \frac{\rho_{k,0}}{\rho_{m,0}} \tag{5.3}$$

$$\Rightarrow \quad \Omega_k(t) = \frac{\Omega_{k,0}}{\Omega_{m,0}} \frac{\Omega_m(t)}{1+z} \tag{5.4}$$

This formula holds back to the matter–radiation equality at $t = t_{\text{eq}}$, where $\Omega_m(t_{\text{eq}}) \approx \frac{1}{2}$ and $z_{\text{eq}} \approx 3000$. Using $\Omega_{k,0}/\Omega_{m,0} \lesssim 10^{-2}$, we find:

$$|\Omega_k(t_{\text{eq}})| \lesssim 10^{-6} \tag{5.5}$$

At earlier times, the universe was radiation-dominated. This can be written as:

$$\frac{\rho_k(t)}{\rho_r(t)} = \frac{\rho_{k,0}}{\rho_{r,0}} \tag{5.6}$$

$$\Rightarrow \quad \Omega_k(t) = \frac{\Omega_k(t_{\text{eq}})}{\Omega_r(t_{\text{eq}})} \frac{(1+z_{\text{eq}})^2}{(1+z)^2} \Omega_r(t) \tag{5.7}$$

During Big Bang Nucleosynthesis (BBN), with $z \approx 4 \times 10^8$, the curvature must satisfy:

$$|\Omega_k(t_{\text{BBN}})| \lesssim 10^{-16} \tag{5.8}$$

At the time of the electroweak phase transition ($z \approx 10^{15}$), the curvature must be:

$$|\Omega_k(t_{\text{EW}})| \lesssim 10^{-30} \tag{5.9}$$

This extreme fine-tuning of Ω near unity is known as the **flatness problem**.

—

The **horizon problem** refers to the observed homogeneity and isotropy of the universe on very large scales. The CMB has a uniform temperature of 2.725 K in all directions. However, in the standard Hot Big Bang scenario, different regions of the sky were outside each other's particle horizons when the CMB formed — meaning they could not have been in causal contact to thermalize.

For a purely matter-dominated universe, with $a(t) = (t/t_0)^{2/3}$, the particle horizon at time

t is defined by:

$$d_H(t) = c\,a(t) \int_0^t \frac{dt'}{a(t')} = 3ct \tag{5.10}$$

We can use $H(t) = \frac{2}{3t} = H_0\,a(t)^{-3/2}$ to express this in terms of redshift:

$$d_H(z) = \frac{2cH_0^{-1}}{(1+z)^{3/2}} \tag{5.11}$$

The particle horizon today is $d_H(t_0) = 2cH_0^{-1}$. The distance $d_H(z)$ subtends an angle on the sky given by:

$$\theta \approx \frac{(1+z)d_H(z)}{d_H(t_0)} \approx \frac{1}{\sqrt{1100}} \approx 0.03 \text{ rad} \quad \Rightarrow \quad \theta \approx 1.7^\circ \tag{5.12}$$

Thus, patches of the sky separated by more than about 1.7° had no causal contact at the time of CMB formation. How different regions of the universe reached the same temperature without ever interacting — that is the **horizon problem**.

5.2 The Inflation Solution

Inflation can most generally be defined as the hypothesis that there was a period early in the history of the universe when the expansion was accelerated outward — an epoch when

$$\ddot{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) \tag{5.13}$$

tells us that $\ddot{a} > 0$ when $P < \varepsilon/3$. This would have taken place if the universe were dominated by a component with equation-of-state parameter $w < -1/3$.

An inflationary phase naturally solves the flatness problem. Such a phase must be driven by some background energy density that scales as

$$\rho_{\rm inf} \approx \frac{1}{a^{2/n}} \tag{5.14}$$

which, for $n > 1$, clearly dilutes away more slowly than the curvature term $\rho_k \propto 1/a^2$. During the inflationary phase, the curvature energy density ρ_k is driven to zero, as are the matter (ρ_m) and radiation (ρ_r) densities.

To resolve the flatness, horizon, and monopole problems, we assume that the universe underwent a period of exponential expansion, with

$$a(t) \approx e^{H_{\text{inf}} t} \tag{5.15}$$

sometime during its early radiation-dominated phase. For simplicity, we suppose that inflation lasts for some time T. If, prior to inflation, the physical horizon had size d_I, then by the end of inflation this region of space has been blown up to

$$d_F = e^{H_{\text{inf}} T}\, d_I \tag{5.16}$$

The amount of inflation is quantified by

$$N = H_{\text{inf}} T \tag{5.17}$$

If the inflation occurred at redshift z_{inf}, then

$$d_{\text{now}} = e^N (1 + z_{\text{inf}}) d_I \tag{5.18}$$

The whole point of inflation is to ensure that this length d_{now} is much larger than what we can see in the sky.

After inflation, the expansion of the universe was first dominated by radiation, with $H \propto 1/a^2$, and later by matter, with $H \propto 1/a^{3/2}$. Even though the majority of cosmic time is in the matter-dominated era, most of the expansion actually takes place during the radiation-dominated epoch when energy densities were much higher. We can write

$$\frac{H_{\text{inf}}}{H_0} \approx (1 + z_{\text{inf}})^2 \tag{5.19}$$

Thus, we have

$$e^N > \left(\frac{H_{\text{inf}}}{H_0}\right)^{1/2} = z_{\text{inf}} \tag{5.20}$$

The general expectation is that inflation took place at a time corresponding to redshift $z_{\text{inf}} \approx 10^{15}$, giving

$$z_{\text{inf}} > 10^{15} \quad \Rightarrow \quad N > 35 \tag{5.21}$$

If inflation occurred at $z_{\text{inf}} \approx 10^{15}$, then the Hubble scale during inflation was

$$H_{\text{inf}} \approx 10^{12}\ \text{s}^{-1} \tag{5.22}$$

In this case, inflation lasted only

$$T \approx 10^{-11}\ \mathrm{s} \tag{5.23}$$

Many models posit that inflation took place at an epoch when the early universe approached Planckian energy scales:

$$z_{\mathrm{inf}} \approx 10^{27} \quad \Rightarrow \quad N > 62 \tag{5.24}$$

In this case,

$$H_{\mathrm{inf}} \approx 10^{36}\ \mathrm{s}^{-1}, \qquad T \approx 10^{-35}\ \mathrm{s} \tag{5.25}$$

which is an extraordinarily short time scale.

During a period when the universe was expanding exponentially ($a \propto e^{H_{\mathrm{inf}}t}$), the number density of monopoles — if they were neither created nor destroyed — decreased exponentially:

$$n_M \propto e^{-3H_{\mathrm{inf}}t} \tag{5.26}$$

For instance, if inflation started around the GUT epoch, when the number density of magnetic monopoles was

$$n_M(t_{\mathrm{GUT}}) \approx 10^{82}\ \mathrm{m}^{-3}, \tag{5.27}$$

then after the additional expansion from $a(t_f) \approx 2 \times 10^{-27}$ to $a_0 = 1$, the present-day number density would be

$$n_M(t_0) \approx 2 \times 10^{-83} \approx 5 \times 10^{-16}\ \mathrm{Mpc}^{-3}. \tag{5.28}$$

With such an extremely small density, there would likely be no monopoles at all within the last scattering surface.

5.3 The Inflaton Field

Inflation explains several puzzling aspects of cosmology, but questions remain: What triggers inflation at $t = t_i$, and what turns it off at $t = t_f$? If inflation reduces the number density of monopoles to unpredictably low levels, why doesn't it also reduce the number density of photons to similarly low levels? Why doesn't inflation flatten out the local curvature due to fluctuations in the energy density, just as it flattens out the global curvature?

The effective way to implement a transient, inflationary phase in the early universe is to posit the existence of a new field, denoted by $\phi(x,t)$, a scalar field also known as the inflaton field. The dynamics of this scalar field are best described using the action principle. In particle mechanics, the action is an integral over time, but for fields, the action is an integral over both space and time.

If ϕ has units of energy and its potential $V(\phi)$ has units of energy density, then the energy density of the inflation field is written as

$$\varepsilon_\phi = \frac{1}{2\hbar c^3}\dot{\phi}^2 + V(\phi) \tag{5.29}$$

in a region of space where ϕ is homogeneous. The pressure of the inflation field is

$$P_\phi = \frac{1}{2\hbar c^3}\dot{\phi}^2 - V(\phi). \tag{5.30}$$

In Minkowski spacetime, the action takes the form

$$S = \int d^3x\, dt \left[\frac{1}{2}\dot{\phi}^2 - \frac{c^2}{2}\nabla\phi\cdot\nabla\phi - V(\phi)\right], \tag{5.31}$$

where for simplicity, we take the potential as

$$V(\phi) = \frac{1}{2}m^2\phi^2, \tag{5.32}$$

with m being the mass associated with the field. This action is the field-theoretic analogue of the simple harmonic oscillator.

The equation of motion for ϕ follows from the principle of least action. Under variation $\phi \to \phi + \delta\phi$, the action changes as

$$\begin{aligned}\delta S &= \int d^3x\, dt \left[\dot{\phi}\,\delta\dot{\phi} - c^2\nabla\phi\cdot\nabla\delta\phi - \frac{\partial V}{\partial\phi}\,\delta\phi\right] \\ &= \int d^3x\, dt \left[-\ddot{\phi} + c^2\nabla^2\phi - \frac{\partial V}{\partial\phi}\right]\delta\phi. \end{aligned} \tag{5.33}$$

Requiring $\delta S = 0$ for all $\delta\phi$ gives the equation of motion

$$\ddot{\phi} - c^2\nabla^2\phi + \frac{\partial V}{\partial\phi} = 0, \tag{5.34}$$

known as the **Klein–Gordon equation**.

—

We now generalize the action to a scalar field in a homogeneous and isotropic Friedmann–Robertson–Walker (FRW) universe. In flat space, the FRW metric is

$$ds^2 = -c^2 dt^2 + a^2(t)\, d\mathbf{x}^2, \tag{5.35}$$

where the scale factor $a(t)$ changes the spatial distances.

The action now becomes

$$S = \int d^3x\, dt\, a^3(t) \left[\frac{1}{2}\dot{\phi}^2 - \frac{c^2}{2a^2(t)} (\nabla\phi)^2 - V(\phi) \right]. \tag{5.36}$$

As we are only interested in spatially homogeneous solutions, we take $\nabla\phi = 0$ and $\phi(x, t) = \phi(t)$. Then the action simplifies to

$$S = \int d^3x\, dt\, a^3(t) \left[\frac{1}{2}\dot{\phi}^2 - V(\phi) \right]. \tag{5.37}$$

Varying the action gives

$$\begin{aligned} \delta S &= \int d^3x\, dt\, a^3(t) \left[\dot{\phi}\, \delta\dot{\phi} - \frac{\partial V}{\partial \phi}\, \delta\phi \right] \\ &= \int d^3x\, dt \left[-\frac{d}{dt}\left(a^3 \dot{\phi} \right) - a^3 \frac{\partial V}{\partial \phi} \right] \delta\phi. \end{aligned} \tag{5.38}$$

Setting $\delta S = 0$ for all $\delta\phi$ yields the equation of motion:

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0, \tag{5.39}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. The extra term $3H\dot{\phi}$ acts like a friction term and is referred to as **Hubble friction** or **Hubble drag**.

The total energy density and pressure of the scalar field are

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{5.40}$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{5.41}$$

The corresponding Friedmann equation is

$$H^2 = \frac{1}{3M_{\rm pl}^2} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right), \tag{5.42}$$

where $M_{\rm pl}^2 = \dfrac{c^2}{8\pi G}$ is the reduced Planck mass squared.

Taking the time derivative of the Friedmann equation gives

$$2H\dot{H} = \frac{1}{3M_{\rm pl}^2}\left(\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi}\right). \tag{5.43}$$

Using the field equation to eliminate $\ddot{\phi}$, one obtains the **Raychaudhuri equation**:

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3c^2}\left(\dot{\phi}^2 - V(\phi)\right). \tag{5.44}$$

Thus, an inflationary phase occurs when the potential energy dominates the kinetic energy, i.e. $V(\phi) > \dot{\phi}^2$. In the limit $V(\phi) \gg \dot{\phi}^2$, the equation of state approaches $P_\phi \simeq -\rho_\phi$, which mimics a cosmological constant.

We start with a scalar field sitting high on its potential with small $\dot{\phi}$. This leads to inflation. As the scalar rolls down, it gains kinetic energy and the inflationary phase eventually ends. The presence of the Hubble friction term ensures that the scalar can come to rest, rather than oscillating indefinitely.

—

Under the **slow-roll** conditions,

$$V(\phi) \gg \tfrac{1}{2}\dot{\phi}^2, \qquad |\ddot{\phi}| \ll |3H\dot{\phi}|, \tag{5.45}$$

the equations simplify to

$$H^2 \approx \frac{8\pi G}{3c^2}V(\phi), \tag{5.46}$$

$$3H\dot{\phi} \approx -\frac{\partial V}{\partial \phi}. \tag{5.47}$$

The universe ceases to inflate when $V(\phi) \approx \dot{\phi}^2$, which typically occurs at $\phi = \phi_{\rm end}$. By this time, the universe will have expanded by a factor

$$\frac{a(t_{\rm end})}{a(0)} \approx \exp\left[\frac{2\pi G\phi_0^2}{c^2} - \frac{1}{3}\right]. \tag{5.48}$$

Starting the scalar field higher up the potential produces an exponentially larger expansion.

6 Structure Formation

The universe is homogeneous and isotropic only on large scales. On smaller scales, it contains density fluctuations ranging from subatomic quantum fluctuations up to the large superclusters and voids ($\sim 50\,\text{Mpc}$ across) that characterize the distribution of galaxies in space.

HDM(Hot dark matter leads) to structure formation that does not agree with observations. Therefore, we can exclude HDM as the dominant constituent of dark matter. If the only matter in the universe was cold dark matter, then structure formation would be driven solely by gravity. Adding baryonic matter, which is capable of absorbing, emitting, and scattering light, complicates the process of structure formation.

Before the time of decoupling at $z_{\text{dec}} \simeq 1090$, the baryonic matter was tightly coupled to the photons. For $z > 1090$, the interaction of baryonic matter with photons prevented dense baryonic lumps from forming.

The average density of baryonic matter today, at $t_0 \simeq 13.7\,\text{Gyr}$, is

$$\rho_{\text{bary},0} = 4.2 \times 10^{-28}\,\text{kg}\,\text{m}^{-3} = 6.2 \times 10^{9}\,\text{M}_\odot\,\text{Mpc}^{-3}\,. \tag{6.1}$$

We find that the density of stars and interstellar gas is approximately

$$\rho_{\text{sn}} \approx 0.095\,M_\odot\,\text{Mpc}^{-3} \approx 6.4 \times 10^{-21}\,\text{kg}\,\text{m}^{-3}\,. \tag{6.2}$$

This represents an overdensity of

$$\delta_m = \frac{\rho_{\text{sn}} - \rho_{\text{bary},0}}{\rho_{\text{bary},0}} \sim 2 \times 10^{7}\,. \tag{6.3}$$

6.1 The Continuity Equation

To derive the continuity equation, we consider a non-relativistic fluid with mass density ρ, pressure $P \ll \rho$, and velocity $\mathbf{u}$. By denoting the position vector of a fluid element by $\mathbf{r}$ and time by t, the equations of motion are given by basic fluid dynamics.

Mass conservation implies the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla_{\mathbf{r}} \cdot (\rho \mathbf{u}) = 0\,. \tag{6.4}$$

Momentum conservation leads to the Euler equation:

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{r}}\right) \mathbf{u} = -\frac{\nabla_{\mathbf{r}} P}{\rho} - \nabla_{\mathbf{r}} \Phi \,, \tag{6.5}$$

which is simply Newton's second law, $\mathbf{F} = m\mathbf{a}$, for a fluid element. The gravitational potential Φ is determined by the Poisson equation:

$$\nabla^2_{\mathbf{r}} \Phi = 4\pi G \rho \,. \tag{6.6}$$

We decompose all quantities into background values and perturbations:

$$\rho(t, \mathbf{r}) = \bar{\rho}(t) + \delta\rho(t, \mathbf{r}) \,, \tag{6.7}$$

and similarly for the pressure, velocity, and potential.

Static spacetime case:

Ignoring gravity ($\Phi \equiv 0$), the background solution is simply $\bar{\rho} =$ constant and $\bar{\mathbf{u}} = 0$. The linearized evolution equations for the fluctuations are:

$$\frac{\partial \, \delta\rho}{\partial t} = -\bar{\rho} \, \nabla_{\mathbf{r}} \cdot \mathbf{u} \,, \tag{6.8}$$

$$\bar{\rho} \, \frac{\partial \mathbf{u}}{\partial t} = -\nabla_{\mathbf{r}} \delta P \,. \tag{6.9}$$

Combining these two, we find:

$$\frac{\partial^2 \, \delta\rho}{\partial t^2} - \nabla^2_{\mathbf{r}} \delta P = 0 \,. \tag{6.10}$$

For adiabatic fluctuations, the pressure fluctuations are proportional to the density fluctuations:

$$\delta P = c_s^2 \, \delta\rho \,,$$

where c_s is the speed of sound. Then the equation takes the form of a wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2_{\mathbf{r}}\right) \delta\rho = 0 \,. \tag{6.11}$$

A plane wave solves this:

$$\delta\rho = A \, e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})} \,,$$

where $\omega = c_s k$ with $k \equiv |\mathbf{k}|$. Thus, in a static spacetime, fluctuations oscillate with constant amplitude if gravity is neglected.

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Expanding universe:

In an expanding universe, the relation between physical coordinates **r** and comoving coordinates **x** is

$$\mathbf{r}(t) = a(t)\,\mathbf{x}\,. \tag{6.12}$$

The velocity field is then

$$\mathbf{u}(t) = \dot{\mathbf{r}} = H\mathbf{r} + \mathbf{v}\,, \tag{6.13}$$

where **v** is the peculiar velocity.

The spatial derivatives transform as

$$\nabla_{\mathbf{r}} = a^{-1}\nabla_{\mathbf{x}}\,. \tag{6.14}$$

The time derivatives at fixed **r** and **x** are related by

$$\left(\frac{\partial}{\partial t}\right)_{\mathbf{r}} = \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} - H\,\mathbf{x}\cdot\nabla_{\mathbf{x}}\,. \tag{6.15}$$

Substituting these into the continuity equation gives

$$\left[\frac{\partial}{\partial t} - H\,\mathbf{x}\cdot\nabla_{\mathbf{x}}\right]\Big[\bar{\rho}(1+\delta)\Big] + \frac{1}{a}\nabla_{\mathbf{x}}\cdot\Big[\bar{\rho}(1+\delta)(Ha\mathbf{x}+\mathbf{v})\Big] = 0\,. \tag{6.16}$$

The fractional density perturbation is defined as

$$\delta \equiv \frac{\delta\rho}{\bar{\rho}}\,. \tag{6.17}$$

Zeroth order (background):

$$\frac{\partial\bar{\rho}}{\partial t} + 3H\bar{\rho} = 0\,, \tag{6.18}$$

which gives $\bar{\rho} \propto a^{-3}$ the continuity equation for a homogeneous universe.

First order (perturbations):

$$\frac{\partial(\bar{\rho}\delta)}{\partial t} + 3H\bar{\rho}\delta + \frac{\bar{\rho}}{a}\nabla_{\mathbf{x}}\cdot\mathbf{v} = 0\,. \tag{6.19}$$

Using the background equation to eliminate $\dot{\bar{\rho}}$, we obtain

$$\dot{\delta} = -\frac{1}{a}\nabla_{\mathbf{x}}\cdot\mathbf{v}\,. \tag{6.20}$$

The Euler equation becomes

$$\dot{\mathbf{v}} + H\mathbf{v} = -\frac{1}{a\bar{\rho}}\nabla_{\mathbf{x}}\,\delta P - \frac{1}{a}\nabla_{\mathbf{x}}\,\delta\Phi\,. \tag{6.21}$$

In the absence of pressure and gravitational perturbations, this implies $\mathbf{v} \propto a^{-1}$.

Finally, the Poisson equation takes the form

$$\nabla^2_{\mathbf{x}}\,\delta\Phi = 4\pi G a^2 \bar{\rho}\,\delta\,. \tag{6.22}$$

6.2 Density Perturbations

Combining ∂_t with ∇, we find

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2}\nabla^2\delta = 4\pi G\bar{\rho}\,\delta \tag{6.23}$$

This tells us that below the Jeans length, the fluctuations oscillate with decreasing amplitude. Above the Jeans length, the fluctuations experience power-law growth, rather than the exponential growth found in a static space.

The Newtonian framework describes the evolution of matter fluctuations. During the matter-dominated era,

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m = 0 \tag{6.24}$$

Here, we have dropped the pressure term since $c_s = 0$ for linearized CDM fluctuations. Since $a \propto t^{2/3}$, we have $H = \frac{2}{3t}$, and hence

$$\ddot{\delta}_m + \frac{4}{3t}\dot{\delta}_m - \frac{2}{3t^2}\delta_m = 0 \tag{6.25}$$

where we have used $4\pi G\bar{\rho}_m = \frac{3}{2}H^2$. Trying $\delta_m \propto t^p$ gives the following two solutions:

$$\delta_m \propto \begin{cases} t^{-1} \propto a^{-3/2}, & \text{decaying mode} \\ t^{2/3} \propto a, & \text{growing mode} \end{cases} \tag{6.26}$$

Hence, the growing mode of dark matter fluctuations grows linearly with the scale factor during the matter-dominated era.

During the radiation-dominated era, the equation becomes

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\sum_I \bar{\rho}_I\delta_I = 0 \tag{6.27}$$

It is the total density fluctuation $\delta\rho = \delta\rho_m + \delta\rho_r$ which sources $\delta\Phi$. Radiation fluctuations on scales smaller than the Hubble radius oscillate as sound waves, and their time-averaged density contrast vanishes. It follows that CDM is essentially the only clustered component during the acoustic oscillations of the radiation, and so

$$\ddot{\delta}_m + \frac{1}{t}\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m \approx 0 \tag{6.28}$$

Since δ_m evolves only on cosmological timescales, we have

$$\ddot{\delta}_m \sim H^2\delta_m \sim \frac{8\pi G}{3}\bar{\rho}_r\delta_m \gg 4\pi G\bar{\rho}_m\delta_m \tag{6.29}$$

where we have used $\bar{\rho}_r \gg \bar{\rho}_m$. We now find that

$$\delta_m \propto \begin{cases} \text{constant}, \\ \ln t \propto \ln a, \end{cases} \tag{6.30}$$

We see that the rapid expansion due to the effectively unclustered radiation reduces the growth of δ_m to only logarithmic. We need to wait until the universe becomes matter-dominated for the dark matter density fluctuations to grow significantly.

During the Λ-dominated era,

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\sum_I \bar{\rho}_I\delta_I = 0 \tag{6.31}$$

where $I = m, \Lambda$. As far as we can tell, dark energy does not cluster, so we can write

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m = 0 \tag{6.32}$$

In the Λ-dominated regime, $H^2 \approx \text{const.} \gg 4\pi G\bar{\rho}_m$. By dropping the last term, we get

$$\ddot{\delta}_m + 2H\dot{\delta}_m \approx 0 \tag{6.33}$$

which has the following solutions:

$$\delta_m \propto \begin{cases} \text{constant}, \\ e^{-2Ht} \propto a^{-2}, \end{cases} \tag{6.34}$$

We see that the matter fluctuations stop growing once dark energy comes to dominate.

7 Conclusion

This study has outlined the physical framework for understanding the density perturbation in the universe. The Newtonian Friedmann equation provides a powerful tool for modeling the dynamics of cosmic expansion, while anisotropy in the CMB pinpoints the initial density fluctuation, which further attracts more and more matter, much like the Matthew effect. The necessity of a non-baryonic component is demonstrated through the application of the virial theorem to galaxy clusters, thus confirming the gravitational dominance of Dark Matter in structure formation. In addition, the inflation concept does explain the naturalness problem and provides a theoretical basis for the initial conditions leading to the formation of large-scale structures.

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Approval

The internship report titled “Newtonian Structure Formation” submitted by Subaita Rahman Supti, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfillment of the requirements of the internship program. The internship was conducted under the supervision of **Dr. Rafid Mahbub** during the period **15 July 2025** to **15 October 2025**.

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