Fault-Tolerant Quantum Computation from Classical Ising-Type Theories

Project Report

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Abstract

This report summarizes a research project investigating the generalization of the two-dimensional $\mathbb{Z}_2$ Toric Code Model to a d-dimensional $\mathbb{Z}_N$ Toric Code Model. Using a path integral formulation and the Trotter-Suzuki decomposition, we map the quantum partition function $Z = \mathrm{Tr}(e^{-\beta \hat{H}})$ of the d-dimensional lattice to that of an equivalent $(d+1)$-dimensional classical Ising-type lattice. The significance of this mapping lies in its ability to translate the problem of quantum fault tolerance into the language of classical phase transitions. We derive the explicit action and Boltzmann weights of the Ising-type lattice as functions of the original quantum couplings. We then study in detail the special case $N = 2$ and the code limit (zero perturbation, low temperature), and further investigate the associated Kramers-Wannier duality.

1. Introduction

Quantum computation is inherently fragile due to noise and decoherence. To achieve fault tolerance, quantum information must be protected from local errors. Topological quantum error-correcting codes, such as Kitaev's toric code [1], provide a natural way to realize this protection through global degrees of freedom that are resistant to local perturbations. However, this protection is not absolute. Both thermal noise and perturbations (errors in the Hamiltonian) can induce a quantum phase transition, destroying the topological order and the encoded quantum information. Understanding the threshold for this breakdown is very important.

This research project approaches the problem by reformulating it into a more manageable framework- that of classical statistical mechanics. The core idea is from the fact that the d-dimensional quantum system, when subjected to perturbations, is equivalent to a $(d+1)$-dimensional classical system at a finite temperature. Specifically, in this study, we reformulate the Z_N toric code in the language of classical Ising-type theories, use the Trotter-Suzuki decomposition to derive an equivalent $(d+1)$-dimensional classical model, and then study the resulting dualities and trying to identify phase transitions that correspond to error thresholds.

This formulation provides a bridge between quantum error correction, topological order, and classical critical phenomena.

2. Methodology

2.1 The Lattice

Consider a d-dimensional hypercubic lattice Λ where

$$d := p + q\,. \tag{1}$$

We will place our $\mathbb{Z}_N$ spins on the p-cells. We assume that the lattice underlies a finite oriented cubical (CW) complex of dimension d. Then we can consistently assign some orientation to each cell of the lattice. For any p-cell b and $(p-1)$-cell a we define the *incidence number*:

$$\varepsilon(b,a) := \begin{cases} +1 & a \subset \partial b \text{ and the orientation of } a \text{ matches the one induced from } \partial b \\ -1 & a \subset \partial b \text{ but the orientation of } a \text{ is the opposite one} \\ 0 & a \not\subseteq \partial b \end{cases} \tag{2}$$

2.2 The $\mathbb{Z}_N$ Quantum Hamiltonian

We assign a N-dimensional Hilbert space $\mathcal{H}_b := \mathbb{C}^N$ to each p-cell b. The total Hilbert space is defined as the tensor product of all the local Hilbert spaces:

$$\mathcal{H} := \bigotimes_{b \in C_p} \mathcal{H}_b. \tag{3}$$

In each local space, we define the clock operators $\hat{X}$ and $\hat{Z}$ that satisfy

$$\hat{X}\left|i\right\rangle = \left|i+1 \,(\mathrm{mod}\, N)\right\rangle, \qquad \hat{Z}\left|i\right\rangle = \omega^i \left|i\right\rangle, \tag{4a}$$

$$\hat{Z}^r \hat{X}^s = \omega^{rs} \hat{X}^s \hat{Z}^r, \qquad \hat{X}^N = \hat{Z}^N = \hat{1}. \tag{4b}$$

$$\hat{X}^\dagger = \hat{X}^{-1}, \qquad \hat{Z}^\dagger = \hat{Z}^{-1}. \tag{4c}$$

where $\omega = e^{i2\pi/N}$, N-th root of unity and $\left|i\right\rangle$ are the basis states.

For a $(p-1)$-cell a define an operator involving a product over all the p-cells attached to a:

$$\hat{A}_a := \prod_{b \supset a} \hat{X}_b^{(-1)^p \varepsilon(b,a)}. \tag{5}$$

Similarly, for a $(p+1)$-cell c define:

$$\hat{B}_c := \prod_{b \subset c} \hat{Z}_b^{\varepsilon(c,b)}. \tag{6}$$

Using $\hat{A}_a$ and $\hat{B}_c$, we can now define the stabilizer operators and the Hamiltonian:

$$\hat{\mathcal{A}}_a := \frac{1}{N} \sum_{n=0}^{N-1} \hat{A}_a^n, \qquad \hat{\mathcal{B}}_c := \frac{1}{N} \sum_{n=0}^{N-1} \hat{B}_c^n. \tag{7}$$

They are both projectors, and in particular Hermitian. And the Hamiltonian is:

$$\hat{H}_0 := -\sum_{a \in C_{p-1}} J_a \hat{\mathcal{A}}_a - \sum_{c \in C_{p+1}} K_c \hat{\mathcal{B}}_c, \tag{8}$$

$$\hat{H}_1 := -\sum_{b \in C_p} \sum_{n=0}^{N-1} \left(g_b^{(n)} \hat{X}_b^n + h_b^{(n)} \hat{Z}_b^n \right), \tag{9}$$

$$\hat{H} := \hat{H}_0 + \hat{H}_1. \tag{10}$$

Note that (8) is nothing but the specialization of [1, eq. 13] to the $\mathbb{Z}_N$ case, without the constant terms.

The full Hamiltonian $\hat{H} = \hat{H}_x + \hat{H}_z$ is the subject of our study, where $\hat{H}_x$ contains all $\hat{X}$-type operators ($\hat{\mathcal{A}}_a$ and $g_b^{(n)}$) and $\hat{H}_z$ contains all $\hat{Z}$-type operators ($\hat{\mathcal{B}}_c$ and $h_b^{(n)}$).

2.3 Mapping to a Classical Statistical Model

Our primary method is to map the quantum partition function $Z = \mathrm{tr}\left(e^{-\beta \hat{H}}\right)$ to a classical one.

1. **Trotter-Suzuki Decomposition:** We apply the Trotter-Suzuki Decomposition formula to write the partition function in the following way: [2]

$$Z = \mathrm{tr}\left(e^{-\beta \hat{H}}\right) = \lim_{M\to\infty} \mathrm{tr}\left(e^{-\frac{\beta}{M}\hat{H}_x} e^{-\frac{\beta}{M}\hat{H}_z}\right)^M \tag{11}$$

2. **Resolution of Identity:** We choose the basis states as $|s\rangle$ (the eigenbasis of $\hat{Z}_b$ with eigenvalue ω^{s_b}) and insert a resolution of identity $\sum_s |s\rangle\langle s| = \hat{1}$ between each Trotter step.

$$Z = \lim_{M\to\infty} \sum_{s(0)} \cdots \sum_{s(M-1)} \langle s(0)|\hat{U}|s(1)\rangle\langle s(1)|\hat{U}|s(2)\rangle \cdots \langle s(M-1)|\hat{U}|s(0)\rangle. \tag{12}$$

   where we have defined $\hat{U} := \left(e^{-\frac{\beta}{M}\hat{H}_x} e^{-\frac{\beta}{M}\hat{H}_z}\right)$ and to track of where we have inserted certain state, we shall denote the basis vectors inserted between the ith and the $(i+1)$th copy of the operator as $|s(i)\rangle$.

3. **Classical Action:** The next step is to calculate the above matrix elements, multiply them and sum over all the slices of s. This procedure creates a $(d+1)$-dimensional lattice $\underline{\Lambda}$ where the original d-dim lattice is the "spatial" extent and the M steps form a "temporal" direction. Roughly, vertical cells of the (d+1)-dim lattice are "world-volumes" of lower dimensional horizontal cells as they are dragged in the time direction. The summation variables $s(i)_b$ (eigenstates of $\hat{Z}_b$) and new summation variables $n(i)_a$ (arising from matrix elements of $\hat{\mathcal{A}}_a$) become the classical degrees of freedom. We package the $\mathbb{Z}_N$ degrees of freedom s and n into a p-forms ϕ on $\underline{\Lambda}$:

$$\phi \in \mathbb{Z}_N^{\underline{C}_p}, \qquad \phi_{b(i)} := s(i)_b, \qquad \phi_{\underline{a}(i)} := n(i)_a \tag{13}$$

3. Key Findings

3.1 The $(d+1)$-Dimensional Classical Action

We finally find the $(d+1)$-dim classical action is:

$$Z = \sum_{\phi \in \mathbb{Z}_N^{\underline{C}_p}} \left[\prod_{c \in \underline{C}^{\parallel}_{p+1}} W^{\parallel}_c((\mathrm{D}\phi)_c) \prod_{\underline{b} \in \underline{C}^{\perp}_{p+1}} W^{\perp}_{\underline{b}}((\mathrm{D}\phi)_{\underline{b}}) \prod_{b \in \underline{C}^{\parallel}_p} V^{\parallel}_b(\phi_b) \prod_{\underline{a} \in \underline{C}^{\perp}_p} V^{\perp}_{\underline{a}}(\phi_{\underline{a}}) \right]. \tag{14}$$

where the weights and sources are given in the following:

$$W_c^{\|}(x) := \exp\left(\frac{\beta K_c}{M}\delta_N(x)\right) \tag{15a}$$

$$V_b^{\|}(x) := \exp\left(\sum_{n=0}^{N-1}\frac{\beta h_b^{(n)}}{M}\omega^{nx}\right) \tag{15b}$$

$$V_a^{\perp}(x) := \delta_N(x) + \frac{e^{\beta J_a/M}-1}{N} \tag{15c}$$

$$W_b^{\perp}(x) := \sum_{j=0}^{N-1}\left(\frac{1}{N}e^{\sum_{m=0}^{N-1}\frac{\beta g_b^{(m)}}{M}\omega^{mj}}\omega^{(-1)^p jx}\right) \tag{15d}$$

This is a $(d+1)$-dimensional statistical mechanical system with $\mathbb{Z}_N$-valued degrees of freedom on p-cells whose Boltzmann weights are given by W, coupled to sources given by V.

3.2 Code Limit

In the code limit ($g, h \to 0$ and $\beta J/M, \beta K/M \to \infty$), the Boltzmann weights $W^{\perp}$ and $W^{\|}$ become δ_N-functions and both of the source terms ($V^{\perp}$ and $V^{\|}$) become field independent and contribute at most an overall multiplicative factor that does not affect dynamics. The partition function is then

$$Z = \left(\prod_{\underline{a}\in\underline{C}_p^{\perp}}\frac{e^{\beta J_a/M}}{N}\right)\sum_{\phi\in\mathbb{Z}_N^{C_p}}\left[\prod_{c\in\underline{C}_{p+1}^{\|}}\left(1+e^{\beta K_c/M}\delta_N((\mathrm{D}\phi)_c)\right)\prod_{\underline{b}\in\underline{C}_{p+1}^{\perp}}\delta_N((\mathrm{D}\phi)_{\underline{b}})\right]. \tag{16}$$

This finding demonstrates that the classical partition function is dominated by configurations with zero "curvature"—i.e., p-cocycles ($D\phi = 0$).

3.3 Special Case $N = 2$

For the special case $N = 2$, the $\mathbb{Z}_N$ clock operators become the standard Pauli matrices σ_x, σ_z. The Hamiltonian is

$$\hat{H} = -\sum_{a\in C_{p-1}} J_a\hat{\mathcal{A}}_a - \sum_{c\in C_{p+1}} K_c\hat{\mathcal{B}}_c - \sum_{b\in C_p}\left(g_b\hat{\sigma}_x^{(b)} + h_b\hat{\sigma}_z^{(b)}\right) \tag{17}$$

and the corresponding Classical partition function becomes

$$Z = \sum_{\phi\in\mathbb{Z}_N^{C_p}}\left[\prod_{c\in\underline{C}_{p+1}^{\|}}\left(1+\left(e^{\frac{\beta K_c}{M}}-1\right)\delta_2((D\phi)_c)\right)\prod_{\underline{b}\in\underline{C}_{p+1}^{\perp}}\frac{1}{2}e^{\frac{\beta}{M}g_{\underline{b}}}\left(1+e^{\frac{\beta}{M}g_{\underline{b}}(\omega-1)}\,\omega^{(-1)^p(D\phi)_{\underline{b}}}\right)\right. \tag{18}$$

$$\left.\prod_{b\in\underline{C}_p^{\|}}\left(e^{\frac{\beta}{M}h_b\omega^{\phi_b}}\right)\prod_{\underline{a}\in\underline{C}_p^{\perp}}\left(\delta_2(\phi_{\underline{a}})+\frac{e^{\beta J_{\underline{a}}/M}-1}{2}\right)\right] \tag{19}$$

For $N = 2$, $\omega = -1$ and after some algebra, the partition function simplifies as

$$Z = \mathcal{N} \sum_{\substack{\mu_b = \pm 1 \\ \nu_{\underline{a}} = \pm 1}} \exp \left[\sum_{c \in \underline{C}_{p+1}^{\parallel}} R_c \prod_{b \subset c} \mu_b + \sum_{\underline{b} \in \underline{C}_{p+1}^{\perp}} G_{\underline{b}} \, \mu_b(i+1) \mu_b(i) \prod_{\underline{a} \subset \underline{b}} \nu_{\underline{a}} + \sum_{b \in \underline{C}_p^{\parallel}} H_b \mu_b + \sum_{\underline{a} \in \underline{C}_p^{\perp}} L_{\underline{a}} \nu_{\underline{a}} \right] . \tag{20}$$

We have renamed the couplings as

$$R_c := \frac{\beta K_c}{2M}, \quad H_b := \frac{\beta h_b}{M}, \quad G_{\underline{b}} := \operatorname{arctanh}\left(e^{-2\beta g_{\underline{b}}/M}\right), \quad L_{\underline{a}} := \operatorname{arctanh}\left(e^{-\beta J_{\underline{a}}/M}\right). \tag{21}$$

We can now package μ_b and $\nu_{\underline{a}}$ into a single field ψ_b

$$\psi_b = \begin{cases} \mu_b, & \text{horizontal } p\text{-cell,} \\ \nu_{\underline{a}}, & \text{vertical } p\text{-cell.} \end{cases}$$

Then the $\mu_b(i+1)\mu_b(i) \prod_{\underline{a} \subset \underline{b}} \nu_{\underline{a}}$ term is nothing but the product of ψ, and finally the $\mathbb{Z}_2$ action becomes

$$S(\mu, \nu) := -\frac{1}{\beta} \left[\left(\sum_{c \in \underline{C}_{p+1}^{\parallel}} R_c + \sum_{c \in \underline{C}_{p+1}^{\perp}} G_c \right) \prod_{b \in \partial c} \psi_b + \left(\sum_{b \in \underline{C}_p^{\parallel}} H_b + \sum_{b \in \underline{C}_p^{\perp}} L_b \right) \psi_b \right]$$

which matches with the result of [3].

3.4 Kramers-Wannier Duality

A major finding of this project is the derivation of a Kramers-Wannier duality for the classical action. By applying a character expansion (Fourier transform) to the partition function $Z_{\underline{\Lambda},p}(W, V)$, we map it to a new partition function on the dual lattice $\underline{\Lambda}^{\vee}$.

$$Z_{\underline{\Lambda},p}(W, V) \propto Z_{\underline{\Lambda}^{\vee},d-p}(\theta^* \tilde{V}, \theta^* \tilde{W}) \tag{22}$$

This duality relates the partition function of the original model to that of a *dual* model, where the classical fields are $(d-p)$-forms. We find this duality maps the couplings as:

- Stabilizer strengths are swapped: $J \longleftrightarrow K$
- Perturbation strengths are swapped: $g \longleftrightarrow h$

This powerful result relates the high-temperature (weak coupling) expansion of the original model to the low-temperature (strong coupling) expansion of the dual model. The phase transition, if one exists, occurs at the self-dual point where the model is equivalent to its dual.

4. Discussion and Future Work

This project has successfully established a rigorous mapping between d-dimensional $\mathbb{Z}_N$ quantum codes and $(d+1)$-dimensional classical statistical models. The significance of this mapping is profound. The stability of the fault-tolerant quantum phase is equivalent to the ordered phase of a corresponding classical model. The breakdown of the quantum code is a classical phase transition.

Based on the solid theoretical framework established, the immediate next step is to study the phase transition. We are now trying to do a Monte-Carlo simulation to find the phases of the classical lattice and critical points for those phase transitions.

References

[1] A. Y. Kitaev, *Fault tolerant quantum computation by anyons*, *Annals Phys.* **303** (2003) 2 [`quant-ph/9707021`].

[2] H. F. Trotter, *On the product of semi-groups of operators*, *Proceedings of the American Mathematical Society* **10** (1959) 545.

[3] I. S. Tupitsyn, A. Kitaev, N. V. Prokof'ev and P. C. E. Stamp, *Topological multicritical point in the phase diagram of the toric code model and three-dimensional lattice gauge higgs model*, *Physical Review B* **82** (2010) [`0804.3175`].

Approval

The internship report titled “Fault-Tolerant Quantum Computation from Classical Ising-Type Theories” submitted by **Shanto Chakroborty**, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfillment of the requirements of the internship program.

The internship was conducted under the supervision of **Dr. Nafiz Ishtiaque** during the period **15 July 2025** to **15 October 2025**.

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