PENROSE SINGULARITY THEOREM APPLIED TO KRUSKAL-SZEKERES MANIFOLD

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Chapter 1

Penrose Singularity Theorem

1.1 Preliminaries

We begin with recalling some rudimentary notions from differential geometry following [9, 6].

Definition 1 (Manifold). *An n-dimensional, C^∞, real manifold $\mathscr{M}$ is a set together with the collection of subsets $\{O_\alpha\}$ satisfying the properties:*
1. Each $q \in \mathscr{M}$ lies in at least one O_α i.e. $\{O_\alpha\}$ cover the entire manifold $\mathscr{M}$.
2. For each α, $\exists$ a one-to-one and onto map $\psi_\alpha : O_\alpha \to U_\alpha$ where U_α is an open subset of $\mathbb{R}^n$.
3. If any two sets $O_1 \cap O_2$ where $\alpha = 1, 2$ overlap such that $O_1 \cap O_2 \neq \emptyset$ then we can consider the map $\psi_2 \circ \psi_1^{-1}$ takes the points in $\psi_1[O_1 \cap O_2] \subset U_1 \subset \mathbb{R}^n$ to the points in $\psi_2[O_1 \cap O_2] \subset U_2 \subset \mathbb{R}^n$. These subsets of $\mathbb{R}^n$ have to be open and continuous and the map has to be infinitely continuously differentiable.

Intuitively, a manifold can be thought of as pieces of open subsets that can be "sewn together" smoothly.

Definition 2 (Tangent Space). *The set of all tangent vectors at a point p in the $\mathscr{M}$ manifold forms an n-dimensional vector space called the tangent space $T_p(\mathscr{M})$.*

Definition 3 (Lorentzian Manifold). *A Lorentzian manifold is a pair $(\mathscr{M}, g)$ where M is a differentiable manifold and g is Lorentzian metric tensor field. On Lorentzian manifold a non-zero vector $u \in T_P(\mathscr{M})$ is timelike if $g(u, u) < 0$, null/lightlike if $g(u, u) = 0$ and spacelike if $g(u, u) > 0$.*

In this paper in the further discussions we refer to Lorentzian manifolds by simply calling them 'manifolds'.

Definition 4 (Geodesic). *A geodesic on a manifold $(\mathscr{M}, g)$ is a curve $\gamma : (p, q) \to \mathscr{M}$ whose tangent vector T is parallel-propagated along the curve itself, i.e. a curve which satisfies the equation*

$$\nabla_T T = 0,$$

where ∇_T is the Levi-Civita covariant derivative with respect to T.

In a coordinate system on $\mathscr{M}$, a geodesic is a solution to the geodesic equation $\dot{T}^\mu = -\Gamma^\mu_{\alpha\beta} T^\alpha T^\beta$ where Γ is the Christoffel symbol and T^μ are the components of T in the coordinate basis and $T^\mu = \frac{dx^\mu}{dt}$ such that

$$\ddot{x}^\mu = -\Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta$$

with provided initial values $x(0)$ and $\dot{x}(0)$ where γ is parametrized by the affine parameter λ.
Intuitively, a geodesic is the straightest possible line or a line which curves as little as possible.

Definition 5 (Complete Geodesic). *As in the terminology of Definition 4, a geodesic is **future complete** if $q = \infty$. Similarly, it is **past complete** if $p = -\infty$.*

Definition 6 (Inextendible Geodesic). *A geodesic is **future inextendible** if its future limit, $\lim_{\lambda \to q^-} \gamma(\lambda)$ does not exist in $\mathscr{M}$. Likewise, a geodesic is **past inextendible** if its past limit, $\lim_{\lambda \to p^+} \gamma(\lambda)$, does not live in $\mathscr{M}$.*

Definition 7 (Incomplete Geodesic). *A geodesic is **future incomplete** iff it is future inextendible and $q < \infty$. Similarly, it is **past incomplete** iff it is past inextendible and $p > -\infty$. A geodesic is incomplete if it is either future incomplete or past incomplete or both.*

1.1.1 Cauchy Hypersurface

A Cauchy hypersurface is a 3-dimensional slice of a 4-dimensional spacetime. It can be thought of as snapshots of spacetime at some fixed time.

Definition 8. *For $\Sigma \subset \mathscr{M}$, Σ is a Cauchy hypersurface in $\mathscr{M}$ if every inextendible timelike curve in $\mathscr{M}$ intersects Σ only once.*

An example of a Cauchy hypersurface would be the Minkowski spacetime with coordinates (t,x,y,z) where $t = constant$. If we take $t = tanhx$ on this Minkowski spacetime, it would give us slices that are not Cauchy hypersurfaces.

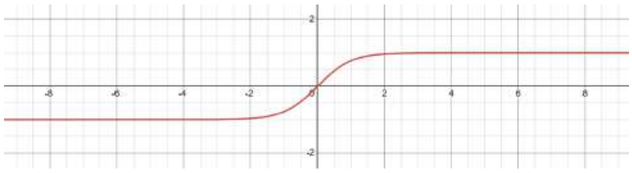

Figure 1.1: Graphical Representation of tanh x

We see that there exists inextendible timelike curves that do not intersect the slice, so it is not Cauchy hypersurface.

1.1.2 Non-total Imprisonment

A spacetime is said to be non-totally imprisoning if there exists no inextendible future causal curve that remains forever trapped in a compact region of spacetime.
Let us take the example of a black hole. We can think of it as a compact region and not even light can escape it. That means that a black hole is an example of a spacetime region that is totally imprisoning. Non total imprisonment ensures that any causal curve cannot be forever trapped in any finite compact region.

1.1.3 Notations and Relations

$a \ll b$ when a and b are points in a chronological curve I
$a \leq b$ when a and b are points in a causal curve J
$a < b$ when a and b are points in causal curve and $a \neq b$; more specifically, a is in the causal past of b
$a \to b$ when a and b belong to the light-like boundary $\mathcal{E}$

Let $\mathscr{M}$ be a Lorentzian manifold. Consider $A \subset \mathscr{M}$.

Definition 9 (Chronological Past and Future)**.** *The chronological past $I^-(p)$ of a point $p \in \mathscr{M}$ is the set of all points from which one can reach the point p on a past directed timelike curve sourcing from p. The chronological future, denoted by $I^+(p)$, is the set of all points that can be reached from a future directed timelike curve originating from p. For the set A ,*

$$I^-(A) = \{b \in \mathscr{M} : \exists\ a \in A \ such\ that\ b \ll a\},$$
$$I^+(A) = \{b \in \mathscr{M} : \exists\ a \in A \ such\ that\ a \ll b\}$$

Definition 10 (Causal Future)**.** *Causal Future, denoted by $J^+(p)$, is the set of all points that can be reached from p by a future-directed causal curve. For the set A ,*

$$J^+(A) = \{b \in \mathscr{M} : \exists\ a \in A \ such\ that\ a < b\}$$

In simpler words, the causal curve comprises of the chronological/ timelike curve and the lightlike one. (similar definition for causal past) This implies

$$I^+(p) \subset J^+(p) \ and \ I^-(p) \subset J^-(p)$$

Definition 11 (Horismos or Lightlike Boundary)**.** *The horismos relation is the difference* $\mathcal{E} = J \backslash I$ *where* J *is called the causal relation such that* $J = \{(a, b) : \exists$ *a causal curve that connects* a *to* b *or* $a = b\}$ *and* I *is the chronological relation where* $I = \{(a, b) : \exists$ *a timelike curve that connects* a *to* b *or* $a = b\}$*. Therefore,* $\mathcal{E}^+ = J^+ \backslash I^+$ *is the future horismos of a point and* $\mathcal{E}^- = J^- \backslash I^-$ *is the past horismos.*

Definition 12 (Causality Condition)**.** *The causality condition holds at a point* $p \in \mathscr{M}$ *if there are no closed curves through* p *and on a subset* $A \subset \mathscr{M}$ *if it holds for all* $p \in A$*[4].*

Definition 13 (Global Hyperbolicity)**.** *A manifold* $\mathscr{M}$ *is globally hyperbolic if the following conditions hold[1]*

- *the causality condition is satisfied on* $\mathscr{M}$*.*
- *the set* $J(p, q) = J^+(p) \cap J^-(q)$ *is compact where* $p < q$ *such that* $p, q \in \mathscr{M}$ *and* $J(p, q) \subset \mathscr{M}$*.*

1.1.4 Convergence of Outgoing Null Rays

Null Geodesic Completeness:
A manifold $\mathscr{M}$ is (null) geodesically complete if all of its future and past directed null geodesics can be extended infinitely far without them ending abruptly. Null geodesics are parametrized by an affine parameter, and this parameter can run over $\mathbb{R}$.

Null Convergence Condition: states that the Ricci curvature tensor has to be non-negative when applied on any null vector, i.e. $Ric(u, u) \geq 0 \ \forall$ null vector X.

1.1.5 Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = -\frac{1}{3}\theta^2 - \sigma^2 + \omega^2 - Ric(u, u) \tag{1.1}$$

where θ is the expansion of congruences, λ is the affine parameter, σ is a measure of shear, ω expresses rotation , Ric is the Ricci curvature and u is the tangent vector of the geodesic.

Multiple nearby null geodesics collectively form a congruence (a bundle of geodesics). Raychaudhuri equation studies the behavior of these congruences. The expansion θ measures the changes in the cross-sectional area of the congruence. Negative θ implies that the congruence is converging or focusing, and positive θ implies that it is diverging/ expanding. The rotation tensor is zero for light-like geodesics if the congruence is orthogonal to the hypersurface. Then we see that if $Ric \geq 0$, then all three terms contribute negatively to the equation making $\frac{d\theta}{d\lambda} \leq 0$. This means that the rate of expansion of congruence with respect to the affine parameter is always decreasing. If θ is initially positive, portraying expansion, the rate decreases until $\theta \to -\infty$ where it approaches the focal point. Einstein's field equation relates Ricci curvature tensor to the energy-momentum tensor implying that matter and energy produces gravitational pull which causes light-like geodesics to converge to a focal point.

1.1.6 Trapped Surfaces

A trapped surface is a spacelike compact codimension 2 submanifold that is future converging [4]; see e.g. [3, 7]. This denotes not only the ingoing rays but the outgoing rays are also future converging, i.e. $\frac{d\theta}{d\lambda}$ from the Raychaudhuri equation for both families of ingoing and outgoing null congruences are negative.

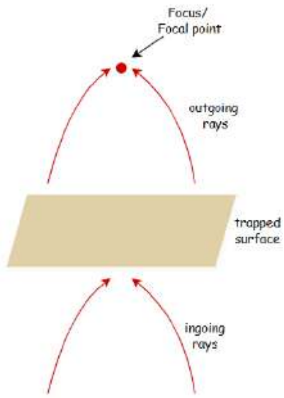

Figure 1.2: Trapped surface

The outgoing geodesics converge to a focus. This focal point is the end of space and time. If the geodesic could have continued after the focal point, then ABD would not be the shortest length or path from A to D anymore. Similarly, as the path goes more and more beyond the focal point, there will always exist a shorter path. So, this ABD curve will no longer even be the geodesic.

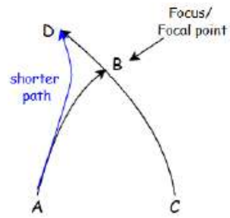

Figure 1.3: The geodesic no longer remains a geodesic if it crosses the focal point. AB=BC

In other words, this geodesic can not be extended infinitely far in the affine parameter, and the geodesic ends abruptly at the focal point, giving rise to geodesic incompleteness.
Any closed surface inside the black hole is a trapped surface. Hence, Penrose proved that every black hole contains a singularity in it-a piece of infinite gravity.

Connected Set:
A connected set is a set that cannot be divided into two non-empty, open subsets such that they do not contain any element in common. A set is connected if there exists no subset which is simultaneously open and closed unless it is empty or the entire set itself.

1.1.7 Penrose Theorem

For a globally hyperbolic spacetime $(\mathscr{M}, g)$ if there is a non-compact hypersurface, then

- There does not exist any non-empty compact set S which is future trapped, i.e. $E^+(S)$ is compact, or such that $\bar{E}^+(S)$ is compact. This means even if set S on the hypersurface is compact, its future horismos and closure of the future horismos are non-compact[3].
- And for the sets that are non-empty and compact, say set S, there exists a future light-like ray emanating from S and entirely contained in $E^+(S)$. Then again, as per the previous point $E^+(S)$ is non-compact. This means that the future light-like ray continues indefinitely in the future direction without getting into the time-like region $I^+(S)$[3].

Theorem 1. *Consider $(\mathscr{M}, g)$ is a globally hyperbolic spacetime with a non-compact Cauchy hypersurface Σ where the null convergence condition holds, i.e. $Ric(u, u) \geq 0$ for all null vectors u on $\mathscr{M}$, and $\mathscr{M}$ contains a trapped surface, then the manifold $\mathscr{M}$ is future null incomplete.*

This theorem is originally due to Roger Penrose [5]. Here, we present a proof following [7].

Proof Consider a compact set S such that $S \subset \mathscr{M}$ where $\mathscr{M}$ is the globally hyperbolic manifold. Firstly, we want to prove that $J^+(S)$ is closed in $\mathscr{M}$. Let (a_n) be a sequence such that $a_n \subset J^+(S)$ and $(a_n) \to a \in \mathscr{M}$. Then $\exists\ b_n$ in S such that $b_n \leq a_n\ \forall n$. Since S is compact, there exists a convergent subsequence (b_m) such that $(b_m) \to b \in S$. Similarly, let (a_m) be a convergent subsequence of (a_n) such that $(a_m) \to a$, then $b_m \leq a_m\ \forall m$ and $b \leq a$. This means that $a \in J^+(b) \subset J^+(S)$. So, $(a_m) \in S \subset J^+(S)$ and $a \in J^+(S)$ i.e. both the sequence and its limit point are within the set $J^+(S)$. This implies that the set $J^+(S)$ is closed.

$E^+(S)$ is an achronal topological hypersurface and S is future trapped; this means that $E^+(S)$ is compact.

Next we want to show that the compactness of $E^+(S)$ contradicts the non-compactness of Cauchy hypersurface. Let $p : \mathscr{M} \to \Sigma$ be a homeomorphic, open, continuous map. We now restrict the domain to $E^+(S)$ only. Since $E^+(S)$ is compact, the image $\rho = p(E^+(S))$ is also compact. A set is said to be compact when it is closed and bounded. This denotes ρ is a closed subset of Σ.

Since $E^+(S) \subset \mathscr{M}$ is a topological hypersurface, it is locally Euclidean i.e. locally homeomorphic to an open subset of $\mathbb{R}^n$ for some dimension n. ρ being continuous homeomorphic implies that the image $\rho(E^+(S))$ is also an open subset of $\mathbb{R}$ in Σ.

Since $\rho(E^+(S))$ is non-empty, closed and open simultaneously, we can conclude that ρ is connected and this is only true if $\rho(E^+(S)) = \Sigma$. This denotes that Σ is compact since the image of a compact set through a homeomorphic continuous map is also compact. However, this acts a contradiction to the initial condition of the Cauchy hypersurface being non-compact, which indicates there exists at least one null geodesic which is incomplete.

Chapter 2

Schwarzschild Spacetime

Recall that a spacetime $(\mathscr{M}, g)$ is a time-oriented Lorentzian manifold.

Definition 14 (Kruskal-Szekeres Manifold)**.** *The Kruskal[2]-Szekeres[8] extension of the Schwarzschild manifold $(\mathscr{M}, g)$ is given by*

$$\mathscr{M} := \{(T, X) \in \mathbb{R} \times \mathbb{R} \mid T^2 - X^2 < 1\} \times \mathbb{S}^2, \tag{2.1}$$

$$g := -\frac{32M^3}{r} e^{-\frac{r}{2M}} (dT^2 - dX^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2}$$

Here $M \in \mathbb{R}_+$ is the Schwarzschild mass parameter, r is a positive parameter, and

$$T^2 - X^2 = (1 - \frac{r}{2M})e^{\frac{r}{2M}}$$

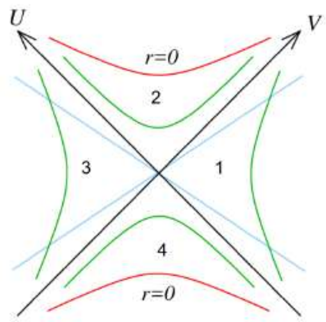

Figure 2.1: The Kruskal diagram for Schwarzschild black hole. U and V axes are the event horizons $r = 2GM$. Red lines are singularities $r = 0$. The green lines in quadrant 1 and 3 represent $r = constant > 2M$ and in quadrant 2 and 4 represent $r = constant < 2M$ and the blue lines represent $t = constant$ [9].

The event horizon is at $r = 2M$. In Kruskal spacetime, it is at $U = 0$ or $V = 0$, given that $U = T - X$ and $V = T + X$. This denotes that the horizon is not one but two surfaces. $V = 0$ is the future horizon- the horizon of black hole (quadrant 2) and $U = 0$ is the past horizon- the horizon of the white hole (quadrant 4). Time goes in the vertical direction as $T = \frac{1}{2}(U + V)$ and space goes in the horizontal direction as $X = \frac{1}{2}(V - U)$. The U and V axes represent the null lines at 45°[9]. Singularity sits at $r = 0 \Rightarrow UV = 1$. The hyperbola $UV = 1$ has two disconnected elements; $U, V > 0$ corresponds to the black hole singularity and $U, V < 0$ corresponds to white hole singularity. This shows singularity inside a black hole is not a point; it is spacelike. Singularity is something that lies in the future. Similarly, for the white hole singularity lives in the past[9].

2.1 Penrose Theorem for Kruskal-Szekeres Manifold

In this subsection, we will be checking if the Penrose theorem works in the Kruskal-Szekeres spacetime. $\mathscr{M}$ is time-orientable- defined by the Kruskal time T which is applicable globally throughout the entire manifold. Maximally extended Schwarzschild solution can be divided into four regions as shown in Figure 2.1 which are entirely covered by the Kruskal-Szekeres coordinates.

Let us take the spacelike hypersurface $T = T_0$ where this surface passes through quadrant I, II and III. Every inextendible timelike curve intersects this surface only once; i.e. it is achronal. We are taking a globally constant Kruskal time. This means that in the 4 dimensional spacetime, it is a 3-dimensional surface with coordinates (X, θ, ϕ). Hence, it is a Cauchy hypersurface. The entire spacetime can be foliated by these Cauchy hypersurfaces. This means that the manifold $\mathscr{M}$ is globally hyperbolic. No causal loops are possible here.

Next, we need to examine if $\mathscr{M}$ satisfies the null convergence condition. From the Kruskal metric,

$$L = -\frac{32M^3}{r}e^{-\frac{r}{2M}}\dot{T}^2 + \frac{32M^3}{r}e^{-\frac{r}{2M}}\dot{X}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 \tag{2.3}$$

For ϕ:

$$\frac{\partial L}{\partial \phi} = 0 \tag{2.4}$$

and

$$\frac{\partial L}{\partial \dot{\phi}} = 2r^2\sin^2\theta\dot{\phi} \tag{2.5}$$

$$\frac{d}{d\tau}(\frac{\partial L}{\partial \dot{\phi}}) = 4r\sin^2\theta\dot{\phi}(\frac{\partial r}{\partial T}\frac{dT}{d\tau} + \frac{\partial r}{\partial X}\frac{dX}{d\tau}) + 4r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + 2r^2\sin^2\theta\ddot{\phi} \tag{2.6}$$

$$\frac{d}{d\tau}(\frac{\partial L}{\partial \dot{\phi}}) - \frac{\partial L}{\partial \phi} = 0 \tag{2.7}$$

$$\Rightarrow 4r\sin^2\theta\dot{\phi}(\frac{\partial r}{\partial T}\frac{dT}{d\tau} + \frac{\partial r}{\partial X}\frac{dX}{d\tau}) + 4r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + 2r^2\sin^2\theta\ddot{\phi} = 0 \tag{2.8}$$

$$\Rightarrow \ddot{\phi} = -\frac{2}{r}\frac{\partial r}{\partial T}\dot{T}\dot{\phi} - \frac{2}{r}\frac{\partial r}{\partial X}\dot{X}\dot{\phi} - 2\cot\theta\dot{\theta}\dot{\phi} \tag{2.9}$$

This gives us the following Christoffels:

$$\Gamma^{\phi}_{T\phi} = -\frac{8M^2T}{r^2}e^{-\frac{r}{2M}} \tag{2.10}$$

$$\Gamma^{\phi}_{X\phi} = \frac{8M^2X}{r^2}e^{-\frac{r}{2M}} \tag{2.11}$$

$$\Gamma^{\phi}_{\theta\phi} = \cot\theta \tag{2.12}$$

For θ:

$$\frac{\partial L}{\partial \theta} = 2r^2\sin\theta\cos\theta\dot{\phi}^2 \tag{2.13}$$

$$\frac{d}{d\tau}(\frac{\partial L}{\partial \dot{\theta}}) = 4r\frac{\partial r}{\partial T}\frac{dT}{d\tau}\dot{\theta} + 4r\frac{\partial r}{\partial X}\frac{dX}{d\tau}\dot{\theta} + 2r^2\ddot{\theta} \tag{2.14}$$

$$\Rightarrow \ddot{\theta} = \sin\theta\cos\theta\dot{\phi}^2 - \frac{2}{r}\frac{\partial r}{\partial T}\dot{T}\dot{\theta} - \frac{2}{r}\frac{\partial r}{\partial X}\dot{X}\dot{\theta} \tag{2.15}$$

which gives us the following:

$$\Gamma^{\theta}_{\phi\phi} = -\frac{\sin 2\theta}{r} \tag{2.16}$$

$$\Gamma^{\theta}_{T\theta} = -\frac{8M^2T}{r^2}e^{-\frac{r}{2M}} \tag{2.17}$$

$$\Gamma^{\theta}_{X\theta} = \frac{8M^2X}{r^2}e^{-\frac{r}{2M}} \tag{2.18}$$

For T:

$$\frac{\partial L}{\partial T} = \frac{32M^3}{r}(\frac{1}{r} + \frac{1}{2M})e^{-\frac{r}{2M}}(\dot{T}^2 - \dot{X}^2)\frac{\partial r}{\partial T} + 2r\dot{\theta}^2\frac{\partial r}{\partial T} + 2r\sin^2\theta\dot{\phi}^2\frac{\partial r}{\partial T} \tag{2.19}$$

$$\frac{d}{d\tau}(\frac{\partial L}{\partial \dot{T}}) = \frac{32M^3}{r}(\frac{1}{r} + \frac{1}{2M})e^{-\frac{r}{2M}}(\frac{\partial r}{\partial T}\dot{T} + \frac{\partial r}{\partial X}\dot{X})2\dot{T} - \frac{32M^3}{r}2\ddot{T} \tag{2.20}$$

$$\Rightarrow \ddot{T} = \frac{1}{2}(\frac{1}{r} + \frac{1}{2M})\frac{\partial r}{\partial T}(\dot{X}^2 + \dot{T}^2) - \frac{r^2}{32M^3}\frac{\partial r}{\partial T}e^{\frac{r}{2M}}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + (\frac{1}{r} + \frac{1}{2M})\frac{\partial r}{\partial X}\dot{X}\dot{T} \tag{2.21}$$

which gives us the following:

$$\Gamma^{T}_{XX} = \frac{4M^2T}{r}(\frac{1}{r} + \frac{1}{2M})e^{-\frac{r}{2M}} \tag{2.22}$$

$$\Gamma^{T}_{TT} = \frac{4M^2T}{r}(\frac{1}{r} + \frac{1}{2M})e^{-\frac{r}{2M}} \tag{2.23}$$

$$\Gamma^{T}_{\theta\theta} = \frac{-rT}{4M} \tag{2.24}$$

$$\Gamma^{T}_{\phi\phi} = \frac{-rT\sin^2\theta}{4M} \tag{2.25}$$

$$\Gamma^{T}_{XT} = -\frac{4M^2X}{r}(\frac{1}{r} + \frac{1}{2M})e^{-\frac{r}{2M}} \tag{2.26}$$

For X:

$$\frac{\partial L}{\partial X} = \frac{32M^3}{r}(\frac{1}{r} + \frac{1}{2M})e^{-\frac{r}{2M}}\frac{\partial r}{\partial X}(\dot{T}^2 - \dot{X}^2) + 2r\frac{\partial r}{\partial X}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \tag{2.27}$$

$$\frac{d}{d\tau}(\frac{\partial L}{\partial \dot{X}}) = -\frac{64M^3}{r}(\frac{1}{r} + \frac{1}{2M})e^{-\frac{r}{2M}}\dot{X}(\frac{\partial r}{\partial T}\dot{T} + \frac{\partial r}{\partial X}\dot{X}) + \frac{64M^3}{r}e^{-\frac{r}{2M}}\ddot{X} \tag{2.28}$$

$$\Rightarrow \ddot{X} = \frac{1}{2}(\frac{1}{r} + \frac{1}{2M})\frac{\partial r}{\partial X}(\dot{T}^2 + \dot{X}^2) + (\frac{1}{r} + \frac{1}{2M})\frac{\partial r}{\partial T}\dot{X}\dot{T} + \frac{r^2}{32M^3}\frac{\partial r}{\partial X}e^{\frac{r}{2M}}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \quad (2.29)$$

giving us the following:

$$\Gamma^X_{TT} = -\frac{4M^2X}{r}(\frac{1}{r} + \frac{1}{2M})e^{-\frac{r}{2M}} \quad (2.30)$$

$$\Gamma^X_{XX} = -\frac{4M^2X}{r}(\frac{1}{r} + \frac{1}{2M})e^{-\frac{r}{2M}} \quad (2.31)$$

$$\Gamma^X_{XT} = \frac{4M^2T}{r}(\frac{1}{r} + \frac{1}{2M})e^{-\frac{r}{2M}} \quad (2.32)$$

$$\Gamma^X_{\theta\theta} = \frac{-rX}{4M} \quad (2.33)$$

$$\Gamma^X_{\phi\phi} = \frac{-rX\sin^2\theta}{4M} \quad (2.34)$$

$$R^T_{\phi\phi T} = \partial_\phi\Gamma^T_{\phi T} - \partial_T\Gamma^T_{\phi\phi} + \Gamma^\phi_{\phi T}\Gamma^T_{\phi\phi} - \Gamma^T_{\phi\phi}\Gamma^T_{TT} - \Gamma^X_{\phi\phi}\Gamma^T_{XT} \quad (2.35)$$

We plug in the above results and simplify the expression to get:

$$R^T_{\phi\phi T} = \frac{r\sin^2\theta}{4M} + M\sin^2\theta(\frac{1}{r} + \frac{1}{2M})e^{-\frac{r}{2M}}(T^2 - X^2) \quad (2.36)$$

Similarly,

$$R^X_{\phi\phi X} = \partial_\phi\Gamma^X_{\phi X} - \partial_X\Gamma^X_{\phi\phi} + \Gamma^\phi_{\phi X}\Gamma^X_{\phi\phi} - \Gamma^X_{\phi\phi}\Gamma^X_{XX} - \Gamma^T_{\phi\phi}\Gamma^T_{TX} \quad (2.37)$$

$$\Rightarrow R^X_{\phi\phi X} = \frac{r\sin^2\theta}{4M} + M\sin^2\theta(\frac{1}{r} + \frac{1}{2M})e^{-\frac{r}{2M}}(T^2 - X^2) \quad (2.38)$$

$$R^\theta_{\phi\phi\theta} = \partial_\phi\Gamma^\theta_{\phi\theta} - \partial_\theta\Gamma^\theta_{\phi\phi} + \Gamma^\phi_{\phi\theta}\Gamma^\theta_{\phi\phi} - \Gamma^T_{\phi\phi}\Gamma^\theta_{T\theta} - \Gamma^X_{\phi\phi}\Gamma^\theta_{X\theta} \quad (2.39)$$

$$\Rightarrow R^\theta_{\phi\phi\theta} = -\sin^2\theta - \frac{2M}{r}\sin^2\theta e^{-\frac{r}{2M}}(T^2 - X^2) \quad (2.40)$$

$$R^\phi_{\phi\phi\phi} = \partial_\phi\Gamma^\phi_{\phi\phi} - \partial_\phi\Gamma^\phi_{\phi\phi} + \Gamma^\eta_{\phi\phi}\Gamma^\phi_{\eta\phi} - \Gamma^\eta_{\phi\phi}\Gamma^\phi_{\eta\phi} \quad (2.41)$$

$$\Rightarrow R^\phi_{\phi\phi\phi} = 0 \quad (2.42)$$

Since $R_{\phi\phi} = R^T_{\phi\phi T} + R^X_{\phi\phi X} + R^\theta_{\phi\phi\theta} + R^\phi_{\phi\phi\phi}$, after adding all components, we obtain

$$R_{\phi\phi} = -\sin^2\theta - \frac{2M}{r}\sin^2\theta e^{-\frac{r}{2M}}(T^2 - X^2) + \frac{2r\sin^2\theta}{4M} + 2M\sin^2\theta(\frac{1}{r} + \frac{1}{2M})e^{-\frac{r}{2M}}(T^2 - X^2) \quad (2.43)$$

Plugging in $T^2 - X^2 = (1 - \frac{r}{2M})e^{\frac{r}{2M}}$,

$$R_{\phi\phi} = -\frac{2M\sin^2\theta}{r} + \frac{r\sin^2\theta}{2M} + 2\frac{M\sin^2\theta}{r}(1 - \frac{r}{2M})(1 + \frac{r}{2M}) = 0 \quad (2.44)$$

Similarly from our calculations we get

$$R_{TT} = R_{XX} = R_{\theta\theta} = 0$$

This implies $R_{\mu\nu} = 0$, i.e. Kruskal spacetime is Ricci flat. Hence, it satisfies the null convergence condition. Next, we consider a sphere $S^2(r \leq 2M)$ where $T = constant$. For the given conditions, we see that $grad\, r < 0$; hence, $\frac{d\theta}{d\lambda} \leq 0$ proving the surface considered is a trapped surface contained in $\mathscr{M}$. See [4] for details. We, therefore, conclude that the Kruskal spacetime $\mathscr{M}$ is future null incomplete.

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Approval

The internship report titled "Penrose Singularity Theorem Applied to Kruskal-Szekeres Manifold" submitted by Samiha Hamid, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfilment of the requirements of the internship program. The internship was conducted under the supervision of **Dr. Onirban Islam** during the period **15 July 2025** to **15 October 2025**.

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