ICTP PWF: Bangladesh Summer Internship Program 2025

Path Integral Approach to QFT for Simple Systems

Report Prepared by

Sabrina Ilham

In collaboration with

Noor A Alam
Samiha Tashin

Supervised by

Dr. Shadman Salam

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1 Introduction

In quantum field theory, the path integral is a formulation that calculates the probability amplitude for a system to transition from one state to another by summing over all possible paths or field configurations of the system weighted by a complex exponential of the classical action for each path.
If we start by the vaccum matrix element $\langle 0; t_f | 0; t_i \rangle$, we get,

$$\langle 0; t_f | 0; t_i \rangle = N \int \mathcal{D}\phi(\bar{x}, t) e^{iS[\phi]}$$

where, $S[\phi] = \int d^4x \, \mathcal{L}[\phi]$ is the time integral going from t_i to t_f. [1] So, the path integral tells us to integrate over all classical field configurations ϕ.

To understand this, we can take the example of the double slit experiment. We see that a single electron leaves the source, heads toward the slits and then strikes the detector. Now, if we try to guess how it got there, we might think that it went through the left slit or the right slit. But quantum mechanics tells us that it went through both which is very strange. In fact, through every possible route consistent with the experiment. The probability for detection is built from a sum of amplitudes over all possible paths. Now, if we think about what would happen if there were three slits or a hundred or a continuous screen filled with openings, we realize, there is no need to count slits at all. The electron's amplitude is given by summing over all imaginable trajectories from source to detector. This is the essence of the path integral.

The idea itself has an older echo in classical wave theory. In the 17th century, Huygens suggested that every point on a wave front behaves like a tiny source of new waves. Fresnel refined this picture later. He realized that the interference patterns arise from adding contributions with the appropriate phases. Feynman's insight was to recognize that quantum mechanics demands the same. Instead of simple phases from distances, the weights come from the action of the system. Each path contributes an oscillating phase proportional to the classical action and quantum behavior emerges from the interference among all these possibilities. [1]

This point of view has an important consequence. In the limit where Planck's constant becomes small, the oscillations cancel out except near the path of stationary action which is nothing but the classical trajectory. Thus, classical mechanics emerges naturally as an approximation to the full quantum picture. In quantum mechanics, all possible paths are realized simultaneously each with its own weight. This is considerably more powerful in quantum field theory. Instead of particles following trajectories, we consider fields fluctuating across spacetime. The transition amplitudes are obtained by integrating over every possible field configuration weighted by the exponential of the classical action. From this formalism, Feynman diagrams emerge naturally, symmetries are handled and Lorentz invariance is also built in from the start.

This simple principle underlies much of modern theoretical physics. While the canonical operator approach is useful in problems like Hydrogen atom model, the path integral has its importance in high energy physics, statistical mechanics and the study of non-perturbative effects such as, instantons or confinement in quantum chromodynamics. It reshapes the way we think about particles and fields not as isolated objects evolving along definite tracks, but as collective outcomes of countless possible histories woven together by interference. The goal of this report is to understand path integral formalism in quantum mechanics and quantum field theory and discuss some applications of it.

2 QM and QFT Approach

Although the path integral in QM and in QFT look similar, the physics behind them is quite different. In quantum mechanics, the path integral is a sum over all possible trajectories of a single particle moving in time.This tells us the probability of the particle going from one point to another. However, in quantum field theory, we are no longer describing just one particle. Instead, the basic object is a field that fills all of the space, and the path integral is a sum over every possible way that this entire field can fluctuate. Because relativistic physics and the creation or annihilation of particles cannot be handled naturally in ordinary quantum mechanics, this shift was necessary. The fields allow the number of particles to change. The path integral over fields automatically respects the symmetries of spacetime such as, Lorentz invariance. In this way, the path integral formulation of QFT is not just a mathematical trick but the natural language for describing a world where particles are excitations of fields and where symmetry principles dictate the form of the laws themselves.

3 Scalar Field Quantization

Using the path integral formalism in quantizing a scalar field generalizes the familiar methods of non-relativistic quantum mechanics to quantum field theory. We start with the analogy between a particle's propagation in space-time and the evolution of configuration of a field through space-time. In non-relativistic quantum mechanics, from an initial position x_i at the time t_i the amplitude for a particle to propagate to final position x_f at time t_f is given by the following matrix element:

$$\langle x_f | e^{-i\hat{H}(t_f - t_i)} | x_i \rangle$$

where we assume $\hbar = 1$ and $\hat{H}$ is the Hamiltonian operator of the system. This amplitude can be re-expressed as a sum over all the possible paths that connect the initial point (x_i, t_i) and the final point (x_f, t_f), which essentially gives rise to the path integral representation of the propagator [2].

In order to construct the path integral, we divide the time interval $[t_i, t_f]$ in equal small discrete segments of $\delta t = (t_f - t_i)/n$ width. And then we insert the complete sets of position eigen-states at each intermediate time step. Each infinitesimal propagation then can be expressed as:

$$\langle x_{j+1} | e^{-i\hat{H}(t_j)\delta t} | x_j \rangle,$$

Here $t_j = t_i + j\delta t$. We can evaluate these matrix elements through introducing complete set of momentum eigen-states which will allow us to convert the operator expressions into Gaussian integrals. The following is the canonical Gaussian integral:

$$\int_{-\infty}^{\infty} dp\, e^{-\frac{1}{2}ap^2+Jp} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}}$$

We use this repeatedly to carry out the momentum integrations that ultimately results in an effective action. This action is expressed in terms of the Lagrangian $L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - V(x,t)$ of the particle. Here, one can explicitly see the Legendre transformation: integrating over the momentum degrees of freedom converts the Hamiltonian form $H(p, x) = \frac{p^2}{2m} + V(x)$ into the Lagrangian form $L(x, \dot{x}) = p\dot{x} - H(p, x)$, yielding $L = \frac{1}{2}m\dot{x}^2 - V(x)$. And this transformation is the centerpiece of the path-integral formulation of the field theory.

In attempts to generalize this to quantum field theory, we replace the particle's position, $x(t)$ by a classical field configuration $\phi(\mathbf{x}, t)$ [1]. The field evolution from the initial state at t_i to a final state at t_f is encapsulated by the vacuum amplitude of the form:

$$\langle 0; t_f | 0; t_i \rangle = \int \mathcal{D}\phi\, e^{iS[\phi]}$$

where $S[\phi] = \int d^4x\, \mathcal{L}[\phi, \partial_\mu \phi]$ is the action functional. And with appropriate boundary conditions, the summation over all intermediate possible field configurations is given by the measure $\mathcal{D}\phi$. More precisely, $\mathcal{D}\phi$ is understood formally as the continuum limit of the discretised product $\prod_x d\phi(x)$, which is not rigorously defined but serves as the functional analogue of the finite-dimensional integration measure. This integral includes contributions from configurations corresponding to multiple-particle states, virtual fluctuations, and disconnected diagrams and thus illustrates the non-perturbative structure of the path integral approach.

The fields in question themselves satisfy canonical commutation relations as they do in the Schrödinger picture,

$$[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})$$

where $\hat{\pi}(\mathbf{x}) = \dot{\hat{\phi}}(\mathbf{x})$ represents the conjugate momentum operator. And the field eigen-states $|\Phi\rangle$ satisfy $\hat{\phi}(\mathbf{x})|\Phi\rangle = \Phi(\mathbf{x})|\Phi\rangle$. Therefore they provide the field-theoretic analogue of the position eigen-states in quantum mechanics. Similarly, due to the eigen-states of the conjugate momentum $\hat{\pi}(\mathbf{x})$, we can insert the complete sets of identity. This allows us to perform the Gaussian integrals analogous to the particle case. The momentum functional integration then converts the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\nabla\hat{\phi})^2 + \frac{1}{2}m^2\hat{\phi}^2$$

into its Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2$$

And therefore, the path integral becomes:

$$\langle 0; t_f | 0; t_i \rangle = \int \mathcal{D}\phi \, \exp\left\{ i \int_{t_i}^{t_f} d^4x \, \mathcal{L}[\phi, \partial_\mu \phi] \right\}$$

Physically, the path integral accounts for the fundamental principle of quantum superposition but applied to fields. It encapsulates the contribution of intermediate history towards the final state of evolution. The classical equations of motion emerge at the limit $\hbar \to 0$ which is known as the stationary phase and provides a deep connection between classical field theory and its quantum counterpart. This formalism also naturally encodes the time-ordered correlation functions. When we insert field operators at specified space-time points, it accounts for their temporal sequence analogous to the canonical operator approach. The $i\epsilon$ prescription in the propagator ensures the correct causal ordering, so that the path integral automatically yields time-ordered products. This inherent time ordering is crucial for the construction of perturbative expansions and also for properly defining Feynman propagators.

4 Time-Ordered Products in QFT

In ordinary quantum mechanics, when we compute transition amplitudes, the central objects are operators acting on a wavefunction. If we want to know how a system evolves, we apply the time evolution operator:

$$U(t) = e^{-i\hat{H}t}$$

If we insert operators at intermediate times, we must be careful to order them properly. This is because quantum operators at different times generally do not commute. So, the order in which they act matters. Time ordering maintains the principle of causality, where effect follows the cause. This ordering has to be imposed 'by hand' through the time-ordering operator T. In the canonical approach, time ordering has to be inserted explicitly:

$$\langle 0 | T\phi(x_1)\phi(x_2) | 0 \rangle$$

But in the path integral formulation, this ordering appears automatically. When we integrate over all field configurations weighted by $e^{iS[\phi]}$, the functional integral is defined with boundary conditions at initial and final times. As we insert fields at intermediate times, the formalism naturally reproduces the time-ordered Green's functions. In other words, path integrals don't need us to manually enforce time ordering, rather it is built into the definition.
Let us insert a field at fixed position and time into the path integral: [1]

$$\mathcal{I} = \int D\phi e^{iS[\phi]\phi(x_j, t_j)} = \int D\phi_1(\bar{x})...D\phi_n(\bar{x}).\langle 0 | e^{-iH(t_n)\delta t} | \phi_n \rangle ... \langle \phi_2 | e^{-iH(t_2)\delta t} | \phi_1 \rangle \langle \phi 1 | e^{-iH(t_n)\delta t} | 0 \rangle \phi_j(\bar{x_j})$$

Now we replace $\phi_j(\bar{x_j})$ by an operator $\hat{\phi}(x_j)$ .We can collapse up all the integrals to give,

$$N\int \mathcal{D}\phi(\bar{x},t)e^{iS[\phi]}\phi(\bar{x_j},t_j) = \langle 0|\hat{\phi}(\bar{x_j},t_j)|0\rangle$$

The fields will be inserted in the appropriate matrix element. In particular, the earlier field will always come out on the right of the later field.
In general, [1]

$$N\int \mathcal{D}\phi(\bar{x},t)e^{iS[\phi]}\phi(x_1)...\phi(x_n) = \langle 0|T\hat{\phi}(x_1)...\hat{\phi}(x_n)|0\rangle$$

This automatic emergence of time ordering has deep physical meaning. It reflects the fact that the path integral sums over complete histories of the field respecting the causal structure of spacetime. Each insertion corresponds to a point in that history.When we average over all possible field configurations, the contribution automatically arranges itself into the time-ordered sequence. Thus, the mathematics aligns perfectly with our physical intuition.

5 Generating Functional

The behavior of Quantum field is characterized by how a disturbance at one spacetime point x_1 influences the field at another point x_2; this is precisely what a **correlation function** measures. However, calculating these functions directly is extremely cumbersome, as the number of possible time-orderings and pairings of fields grows explosively when we consider more points [3].The central insight of the path integral formalism is that all such correlation functions can be derived from a single master object known as the **generating functional**. In mathematics, a generating function, introduced by Abraham de Moivre in 1739, encodes an infinite sequence as the coefficients of a power series. Its exponential form is employed in the path integral formulation of quantum field theory [4, 5].The path integral for a free field theory which computes the vacuum-to-vacuum transition amplitude $Z[0]$, provides a basic but incomplete picture of nature for this very reason. Because our universe is fundamentally interacting, we need a more powerful tool to describe the physical processes of particle creation, propagation, and annihilation that these correlation functions represent. The conceptual leap is achieved by introducing a mathematical device into the path integral itself: an external, classical *source function* $J(x) = J(t, \mathbf{x})$, which acts as a controllable source and sink for the field, thereby constructing the generating functional. A positive value of $J(x)$ at a specific point acts as a "source" that can create a particle, while a negative value acts as a "sink" that can annihilate one [6]. By coupling this source linearly to the quantum field $\phi(x)$ within the action, the path integral is transformed into the **generating functional**:

$$Z[J] = \int \mathcal{D}\phi \, \exp\left(iS[\phi] + i \int d^4x \, J(x)\phi(x) \right) \tag{1}$$

This object, $Z[J]$, no longer represents a simple vacuum amplitude. Instead, it is the vacuum amplitude in the *presence* of the external sources. The expansion of $Z[J]$ systematically organizes physical processes by the number of particles:

$$Z[J] = Z[0] \sum_{s=0}^{\infty} \frac{i^s}{s!} \int d^4x_1 \ldots d^4x_s \, J(x_1) \ldots J(x_s) \, G^{(s)}(x_1, \ldots, x_s), \tag{2}$$

The coefficients $\left(G^{(s)}(x_1, \ldots, x_s) = \langle 0|T\phi(x_1) \ldots \phi(x_s)|0\rangle \right)$ are the s-point Green's functions. Higher-point functions, such as $G^{(4)}$, contain complete information about scattering processes [6]. The most important feature of the generating functional is that, using the path integral formalism, it allows us to systematically derive all the Feynman rules that are usually obtained in canonical quantization. For an interacting theory, the generating functional can be written as

$$Z[J] = \int \mathcal{D}\phi \; e^{i\int d^4x\left[\frac{1}{2}\phi(-\Box - m^2)\phi + J(x)\phi(x)\right]} \; e^{i\int d^4x \frac{g}{3!}\phi^3(x)}, \tag{3}$$

The functional derivatives of Eq. (3) with respect to the source $J(x)$ generate all diagrams and rules equivalent to the Feynman rules in canonical quantization:

$$\begin{aligned} \langle\Omega|T\{\phi(x_1)\ldots\phi(x_n)\}|\Omega\rangle = & \frac{1}{Z[0]} \sum_{k=0}^{\infty} \frac{1}{k!} \left(i\frac{g}{3!} \right)^k \int d^4z_1 \ldots d^4z_k \\ & \langle 0|T\{\phi_0(x_1)\ldots\phi_0(x_n)\phi_0^3(z_1)\ldots\phi_0^3(z_k)\}|0\rangle. \end{aligned} \tag{4}$$

Here, each functional derivative inserts an external field, while each interaction term $\phi^3(z_j)$ acts as a vertex. Contractions between fields automatically generate all propagators, thereby reproducing the complete set of Feynman diagrams and rules [1].

6 Solving The Free Theory

The free scalar field theory is defined by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi,$$

Notice that the Lagrangian being quadratic in the fields makes it exactly solvable within the path-integral formalism. The generating functional $Z_0[J]$ for the free theory is given by:

$$Z_0[J] = \int \mathcal{D}\phi \, \exp\left\{ i \int d^4x \left[-\frac{1}{2}\phi(\Box + m^2)\phi + J(x)\phi(x) \right] \right\}$$

Here, $J(x)$ represents an external source. The action being quadratic reduces the path integral to a Gaussian integral functional and therefore can be evaluated exactly using the known formula:

$$\int \mathcal{D}\phi \, \exp\left[-\frac{1}{2}\phi A \phi + J\phi\right] = (\det A)^{-1/2} \exp\left[\frac{1}{2} J A^{-1} J\right]$$

Here, $A = i(\Box + m^2)$ and A^{-1} is the Green's function, also known as the propagator $\Pi(x-y)$ satisfies $(\Box + m^2)\Pi(x-y) = -\delta^4(x-y)$. Strictly speaking, the factor $(\det A)^{-1/2}$ is an infinite normalization constant, and in practice one defines $Z_0[0] = 1$ so that it can be absorbed. The solution for the propagator in momentum space is [1]

$$\Pi(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}$$

This corresponds to the familiar Feynman propagator that matches with the propagator obtained via canonical quantization using creation and annihilation operators. Therefore, this result confirms the equivalence of the two approaches for free fields.

Following Schwartz's treatment in sections $14.3 - 14.3.1$ of [1], the generating functional $Z_0[J]$ can be used to derive all n-point time-ordered correlation functions:

$$\langle 0|T\{\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\}|0\rangle = (-i)^n \frac{\delta^n Z_0[J]}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0}$$

For two-point functions, we get

$$\langle 0|T\{\hat{\phi}_0(x)\hat{\phi}_0(y)\}|0\rangle = i\Pi(x-y)$$

This demonstrates how the functional derivatives of $Z_0[J]$ generate all free theory correlation functions. Physically, the propagator describes the amplitude for a particle to propagate from y to x that encodes both the causal structure and quantum fluctuations of the field. We can also obtain higher-order n-point functions through similar methods. In interacting theories, the free generating functional provides the basis for perturbative expansions via functional derivatives.

7 Conclusion

The path integral approach provides a unifying way of thinking about quantum mechanics and quantum field theory. This formulation not only makes the connection between classical and quantum physics more transparent but also emphasizes the central role of the action principle. The contrast with quantum mechanics highlights why quantum field theory was necessary in the first place. In non-relativistic QM, causality is not threatened by operator ordering in the same way because, the Hilbert space is global and particle number is fixed. But in QFT, fields spread across spacetime and particles are being created and destroyed. As a result, the local structure of the theory forces time ordering to be fundamental. The path integral provides a framework where this locality and causality are manifest from the start.This is one of the reasons Feynman diagrams arise naturally in the path integral formalism. Each diagram corresponds to a time-ordered sequence of interactions encoded in the perturbative expansion of the generating functional. The operator algebra of QM is replaced by an elegant geometric picture in QFT.

Path integrals demonstrate tunneling in a double-well potential via the **semiclassical approximation in Euclidean time**. This method shows the integral is dominated by classical paths that minimize the action. For the double-well, this identifies a key non-trivial solution called an **instanton**,a specific path connecting the two minima which represents the tunneling event. The total amplitude is a sum over contributions from all such classical solutions, including the trivial path (zero instantons). Summing these contributions reveals a small energy splitting (ΔE) between the lowest energy states. This splitting is the direct physical evidence of tunneling and is a *non-perturbative* effect, invisible to standard perturbation theory.[1]

Moreover, the free theory solution in the path integral approach illustrates the deep connection between the dynamics of classical and quantum field. Its Gaussian form ensures that we can avail exact evaluation of the generating functional and propagators. On the other hand, this formalism emphasizes the summation over all possible field configurations and therefore demonstrates not only how classical equations of motion arise in the stationary phase limit $\hbar \to 0$ but also how it automatically produces the time-ordered correlation functions. These traits crucially set the stage to introduce interactions and to develop both perturbative and non-perturbative methods in quantum field theory.

8 Acknowledgment

We are deeply grateful to our internship mentor, Dr. Shadman Salam, for his continuous guidance, invaluable insights and feedback throughout. We are also grateful to our peers and group members for their active collaboration, constructive engagement and productive discussions that made this experience enjoyable. Finally, we sincerely thank the organizers of the ICTP PWF: Bangladesh Summer Internship Program for giving us this opportunity to explore quantum field theory through the path integral formalism.

9 References

[1] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, Cambridge, 2014.

[2] Ramamurti Shankar. *Principles of Quantum Mechanics*. Plenum Press, New York, NY, USA, 2nd edition, 1994.

[3] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. Addison-Wesley, Reading, USA, 1995.

[4] Wikipedia contributors. Generating function — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Generating_function&oldid=1304941295, 2025. [Online; accessed 28-September-2025].

[5] Wikipedia contributors. Partition function (quantum field theory) — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Partition_function_(quantum_field_theory)&oldid=1305138914, 2025. [Online; accessed 28-September-2025].

[6] Zvi Bern. Quantum field theory in a nutshell. *Physics Today*, 57(4):88–88, 04 2004.

Signature of the Mentor

Dr. Shadman Salam