# Assignment: Hodge Theory

References: Warner ch.6; Evans (2nd ed.); Nicolaescu (May 24, 2025)

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# 1 Problems 2.1 and 2.1.1 — Basics and Consequences

(References: Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Chapter 6.)
We follow Warner's notation and proofs in Chapter 6 where indicated.

## Definitions and preliminary remarks

**Definition 1.1.** Let $M$ be a smooth, oriented, compact Riemannian manifold of dimension $n$. Denote by $\Omega^k(M)$ the space of smooth $k$-forms on $M$.

**Definition 1.2** (Exterior derivative)**.** The exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is the unique degree-1 differential operator characterized by linearity, $d \circ d = 0$, and the Leibniz rule. (Standard; see Warner ch.6.)

**Definition 1.3** (de Rham cohomology)**.** The $k$-th de Rham cohomology is

$$H^k_{\mathrm{dR}}(M) = \frac{\ker(d : \Omega^k \to \Omega^{k+1})}{\operatorname{im}(d : \Omega^{k-1} \to \Omega^k)}.$$

**Definition 1.4** (Riemannian metric and Hodge star)**.** Given a Riemannian metric $g$ on $M$ and an orientation, for each $k$ there is a linear Hodge star operator

$$\star : \Omega^k(M) \to \Omega^{n-k}(M),$$

determined pointwise by the property that for $\alpha, \beta \in \Omega^k(M)$

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \mathrm{dvol}_g,$$

where $\langle \cdot, \cdot \rangle$ is the pointwise inner product induced by $g$ and $\mathrm{dvol}_g$ the Riemannian volume form.

**Definition 1.5** (Adjoint $d^*$ and Laplace–de Rham)**.** The formal $L^2$-adjoint of $d$ is

$$d^* = (-1)^{nk+n+1} \star d \star : \Omega^k(M) \to \Omega^{k-1}(M),$$

and the Laplace–de Rham operator is

$$\Delta = dd^* + d^*d : \Omega^k(M) \to \Omega^k(M).$$

(These formulae and sign conventions follow Warner ch.6; minor sign conventions vary in the literature — we follow Warner for consistency.)

## Hodge theorem (statement)

**Theorem 1.6** (Hodge theorem; Warner ch.6 — statement)**.** *Let $M$ be a compact, oriented, smooth Riemannian manifold. For each $k$,*

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus \operatorname{im} d \oplus \operatorname{im} d^*,$$

*where $\mathcal{H}^k(M) = \{\omega \in \Omega^k(M) : \Delta\omega = 0\}$ is the finite-dimensional space of harmonic $k$-forms. Furthermore, the natural map*

$$\mathcal{H}^k(M) \xrightarrow{\simeq} H^k_{\mathrm{dR}}(M)$$

*sending a harmonic representative to its de Rham cohomology class is an isomorphism.*

(Proof will be developed analytically in Section 2.2 following Evans for the analytic tools; Warner gives the geometric formulation and consequences in ch.6.)

### Consequences (2.1.1) — Hodge decomposition, Poincaré duality, uniqueness and finite-dimensionality

We now state and prove the standard consequences, following Warner's approach and using the Hodge theorem.

**Proposition 1.7** (Hodge decomposition)**.** *For each $k$,*

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus d\Omega^{k-1}(M) \oplus d^*\Omega^{k+1}(M),$$

*and the decomposition is $L^2$-orthogonal.*

*Proof.* Sketch: Using the self-adjointness of $\Delta$ on $L^2$ and the elliptic theory developed in Section 2.2, the operator $\Delta$ has discrete spectrum and the $L^2$-closure of $\operatorname{im}\Delta$ equals $\mathcal{H}^\perp$. Since $\operatorname{im}\Delta = d\Omega^{k-1} \oplus d^*\Omega^{k+1}$, the interior direct-sum decomposition and orthogonality follow. Warner ch.6 gives the topological geometric statement; analytic details are in Section 2.2 below.

**Proposition 1.8** (Harmonic representatives and finite-dimensionality)**.** *Every de Rham cohomology class has a unique harmonic representative. Consequently, $H^k_{\mathrm{dR}}(M) \cong \mathcal{H}^k(M)$ and $\dim H^k_{\mathrm{dR}}(M) < \infty$.*

*Proof.* Existence: choose any representative $\alpha \in \Omega^k$ of the cohomology class. Project $\alpha$ orthogonally onto $\mathcal{H}^k$ using the Hodge decomposition; the projection is closed and represents the same cohomology class. Uniqueness: if $\omega \in \mathcal{H}^k$ is exact, $\omega = d\eta$, then $0 = \langle \omega, \omega\rangle = \langle d\eta, \omega\rangle = \langle \eta, d^*\omega\rangle = 0$, so $\omega = 0$. Finite-dimensionality follows because $\Delta$ is elliptic and self-adjoint with discrete spectrum on compact manifolds (elliptic regularity and spectral theory; see Section 2.2).

**Proposition 1.9** (Poincaré duality via Hodge star)**.** *The Hodge star induces an isomorphism*

$$\star : \mathcal{H}^k(M) \xrightarrow{\cong} \mathcal{H}^{n-k}(M),$$

*and under the identification $\mathcal{H}^k \cong H^k_{\mathrm{dR}}(M)$ this yields Poincaré duality*

$$H^k_{\mathrm{dR}}(M) \cong H^{n-k}_{\mathrm{dR}}(M)^*.$$

*Proof.* If $\omega$ is harmonic then $d\omega = 0$ and $d^*\omega = 0$. Using $\star d = (-1)^{k+1} d^* \star$ one easily checks $\star\omega$ is closed and coclosed, hence harmonic, giving a linear map $\mathcal{H}^k \to \mathcal{H}^{n-k}$. The inverse is $\pm\star$ itself since $\star\star = \pm\mathrm{Id}$. The pairing

$$H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR}}(M) \to \mathbb{R}, \qquad ([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \beta$$

is nondegenerate because for a harmonic representative $\alpha$ the functional $\beta \mapsto \int \alpha\wedge\beta$ corresponds (via $\star$) to the $L^2$-inner product with $\star\alpha$, which is nonzero unless $\alpha = 0$. Thus Poincaré duality follows.

## 2 2.2 — Proof of the Hodge theorem (analytic setup)

(References: *Evans, Partial Differential Equations*, 2nd ed., for Sobolev spaces, elliptic regularity, Lax–Milgram, elliptic estimates.)

### 2.1 Sobolev spaces of differential forms

We briefly set up Sobolev spaces of forms following the standard treatment (see Evans, Ch.5 for scalar Sobolev spaces; extension to vector- and form-valued Sobolev spaces is straightforward).

Let $(M, g)$ be compact smooth Riemannian. Use a finite atlas and a partition of unity to define local coordinate representatives of forms. For $k \in \mathbb{Z}_{\geq 0}$, the Sobolev space $H^k\Omega^r(M)$ is the completion of $\Omega^r(M)$ with respect to the norm

$$\|\omega\|_{H^k}^2 = \sum_{|\alpha| \leq k} \int_M |\nabla^\alpha \omega|^2 \, \mathrm{dvol}_g,$$

where $\nabla$ is any fixed connection (Levi-Civita) and $|\cdot|$ is the pointwise tensor norm. Equivalently use coordinate charts and standard $H^k$ norms on components (Evans Ch.5; Warner uses smooth structures; we follow Evans for functional analysis).

Important properties (Evans Ch.5):

- $H^k\Omega^r(M)$ is a Hilbert space with inner product induced by the $H^k$ norm.
- Continuous inclusions: if $k' > k$ then $H^{k'} \hookrightarrow H^k$.
- For $k > n/2$, Sobolev embedding yields $H^k(M) \hookrightarrow C^0(M)$ (and higher regularity embeddings for tensors/ forms).

### 2.2 Weak formulation and Lax–Milgram

To solve $\Delta\omega = f$ in the space of $k$-forms, write the weak formulation: find $\omega \in H^1\Omega^k(M)$ such that for all test $\eta \in H^1\Omega^k(M)$,

$$\int_M \langle d\omega, d\eta \rangle + \langle d^*\omega, d^*\eta \rangle \mathrm{dvol}_g = \int_M \langle f, \eta \rangle \mathrm{dvol}_g. \tag{2.1}$$

Define the bilinear form $B(\omega, \eta) := \int_M \langle d\omega, d\eta \rangle + \langle d^*\omega, d^*\eta \rangle$. Standard integration by parts and properties of $d, d^*$ show $B$ is symmetric, continuous and coercive on the orthogonal complement of harmonic forms. One then applies the Lax–Milgram theorem (Evans, Ch.6—variational methods) to obtain existence and uniqueness of weak solutions modulo harmonic forms.

### 2.3 Elliptic regularity (sketch of results used)

We invoke the following elliptic regularity result (Evans, Ch.6 and standard elliptic theory):

**Theorem 2.1** (Elliptic regularity — used without proof)**.** *Let $P$ be a uniformly elliptic differential operator of order $m$ with smooth coefficients on a compact manifold $M$. If $u \in H^s$ and $Pu \in H^{s-r}$ (with $r \leq m$ appropriate), then $u \in H^{s+m}$ and the elliptic estimate*

$$\|u\|_{H^{s+m}} \leq C(\|Pu\|_{H^{s-r}} + \|u\|_{H^s})$$

*holds.*

For the Laplace–de Rham $\Delta$, which is elliptic of order 2, this implies that weak $L^2$-solutions are actually smooth (by bootstrapping) and that the kernel of $\Delta$ consists of smooth forms.

### 2.4 Spectral theory and finite-dimensionality

Since $\Delta$ is a nonnegative, symmetric elliptic operator on a compact manifold, the spectrum is discrete with finite-multiplicity eigenvalues tending to infinity. In particular its kernel (harmonic forms) is finite-dimensional. This spectral fact is standard in elliptic theory (Evans Ch.6 sketches; full proofs in classical references).

### 2.5 Putting it together: proof of Hodge theorem (summary)

*Sketch of the Hodge theorem proof.*

1. Use the weak formulation (2.1) and Lax–Milgram to obtain existence of weak solutions of $\Delta u = f$ orthogonal to harmonic forms.
2. Elliptic regularity upgrades weak solutions to smooth ones; hence $\operatorname{im}\Delta$ consists of smooth forms.
3. Spectral decomposition for self-adjoint elliptic operators on compact manifolds gives $L^2$-orthogonal decomposition into kernel and closure of image of $\Delta$. Analytically, $\overline{\operatorname{im}\Delta} = \mathcal{H}^{\perp}$.
4. Because $\operatorname{im}\Delta = d\Omega^{k-1} \oplus d^*\Omega^{k+1}$, we deduce the desired Hodge orthogonal decomposition and the identification $\mathcal{H}^k \cong H^k_{\mathrm{dR}}(M)$.

This completes the proof sketch; details of variational set-up and elliptic regularity are given in Evans (Ch.5–6).

## 3 2.3 — Quick survey of Sobolev theory (Evans, Ch.5)

(References: Evans, Ch.5—Sobolev spaces; Ch.6—elliptic PDE elements used.)

This section collects the main Sobolev facts used in the Hodge-theorem proof.

### 3.1 Sobolev spaces $W^{k,p}$ and $H^k = W^{k,2}$

Definition: for an open set $\Omega \subset \mathbb{R}^n$, $W^{k,p}(\Omega)$ consists of $L^p$-functions whose weak derivatives up to order $k$ are in $L^p$. On compact manifolds use local charts and partitions of unity.

Key properties (Evans Ch.5):

- Density: $C_c^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$ when $\Omega$ is regular.
- Poincaré inequality: on bounded domains, $\|u\|_{L^p} \le C\|\nabla u\|_{L^p}$ for functions with mean zero or suitable boundary conditions.
- Sobolev embeddings: if $kp > n$ then $W^{k,p} \hookrightarrow C^0$; more generally,

  $$W^{k,p}(\Omega) \hookrightarrow W^{\ell,q}(\Omega)$$

  for certain relations between $k, p, \ell, q$ (Rellich–Kondrachov and Sobolev embedding theorems; see Evans Thm. 5.6.1 et seq).

### 3.2 Rellich compactness theorem

**Theorem 3.1** (Rellich–Kondrachov)**.** *If $\Omega$ is bounded with Lipschitz boundary, the embedding $W^{k,p}(\Omega) \hookrightarrow W^{\ell,q}(\Omega)$ is compact provided $k > \ell$ and the exponent relations satisfy the Sobolev embedding scaling conditions (Evans).*

In particular, $H^1(\Omega) \hookrightarrow L^2(\Omega)$ compactly when $\Omega$ is bounded; on compact manifolds similar compactness holds.

### 3.3 Elliptic estimates used in Hodge theorem

For the Laplace–de Rham $\Delta$ (a second-order elliptic operator), Evans provides local and global estimates of the form

$$\|u\|_{H^{s+2}} \le C(\|\Delta u\|_{H^s} + \|u\|_{H^s}),$$

which are the backbone of elliptic regularity and spectral theory used above.

# 4 2.4 — Ellipticity and examples

(References: *Evans* for Sobolev material; but this topic standardly covered in texts on elliptic operators. We follow Evans for ellipticity definitions and basic consequences.)

## 4.1 Principal symbol and ellipticity (definition)

Let $P$ be a linear differential operator of order $m$ acting on sections of a vector bundle (e.g., forms). In local coordinates

$$P = \sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha,$$

and the principal symbol $\sigma_P(x, \xi)$ is the homogeneous polynomial of degree $m$ in $\xi$ defined by the top-order coefficients:

$$\sigma_P(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha.$$

**Definition 4.1.** $P$ is (uniformly) elliptic if for all $x \in M$ and $\xi \in T_x^*M \setminus \{0\}$ the symbol $\sigma_P(x, \xi)$ is invertible as a map between the relevant fibers (e.g., fibers of the bundle on which $P$ acts).

## 4.2 Examples: Laplacian, Hodge Laplacian, Dirac-type operators

**Example 4.2** (Scalar Laplacian)**.** The scalar Laplace–Beltrami operator $\Delta_g$ has principal symbol $\sigma_\Delta(x, \xi) = |\xi|_g^2$, which is positive for $\xi \neq 0$, hence elliptic.

**Example 4.3** (Hodge Laplacian)**.** The Hodge Laplacian $\Delta = dd^* + d^*d$ acting on differential forms has principal symbol $|\xi|^2$ times the identity on the bundle of forms; thus $\Delta$ is elliptic (see Evans and standard PDE references).

**Example 4.4** (Dirac-type operators)**.** Many first-order operators (e.g., the Dirac operator on a spin manifold) satisfy a Clifford relation and have principal symbol satisfying $\sigma_D(x, \xi)^2 = |\xi|^2 \mathrm{Id}$; such operators are elliptic in the sense of first-order elliptic operators.

## 4.3 Basic consequences

Ellipticity implies:

- Regularity: weak solutions are smooth (elliptic regularity).
- Fredholm property: as a map between appropriate Sobolev spaces $P : H^{s+m} \to H^s$ is Fredholm (finite-dimensional kernel and cokernel) on compact manifolds.

These facts are consequences of the parametrix construction (Evans discusses parametrix ideas in Ch.6; complete treatments are in standard analytic texts).

# 5 2.5 — General Hodge theorem (Nicolaescu) and concise proof outline

(Reference: Liviu I. Nicolaescu, *Lectures on the Geometry of Manifolds*, May 24, 2025 — used strictly for this section.)

### 5.1 Statement (general version)

Let $E \to M$ be a smooth vector bundle over a compact oriented Riemannian manifold and let $D$ be an elliptic self-adjoint differential operator of second order built from a connection and the metric (e.g., Laplace-type operator). Then:

1. The space of smooth sections splits orthogonally as $\ker D \oplus \overline{\operatorname{im} D}$.
2. $\ker D$ is finite-dimensional, consisting of smooth sections.

In the special case $E = \Lambda^k T^*M$ and $D = \Delta$ one recovers the classical Hodge theorem.

### 5.2 Concise proof outline (following Nicolaescu)

1. **Ellipticity and parametrix:** For an elliptic operator $D$ there exists a parametrix $G$ (a pseudodifferential operator of order $-2$) such that $GD = \mathrm{Id} - K$ with $K$ smoothing (compact); similarly $DG = \mathrm{Id} - K'$. These are the analytic ingredients giving Fredholm properties.
2. **Fredholm alternative:** Using parametrix, $D$ is Fredholm between Sobolev spaces $H^{s+2} \to H^s$. Self-adjointness gives real spectrum and orthogonal eigenfunction expansion.
3. **Regularity:** If $Du = f$ with $f$ smooth, elliptic regularity implies $u$ is smooth.
4. **Spectral decomposition and orthogonal splitting:** Combine spectral theory and parametrix to obtain the desired orthogonal decomposition $L^2 = \ker D \oplus \overline{\operatorname{im} D}$ and finite-dimensional kernel.

(Precise operator-theoretic details and parametrix/pseudodifferential constructions are treated in Nicolaescu's notes; here we cite those constructions.)

## 6 2.6 — Applications: conformal method and Corvino's lectures

(Reference: Nicolaescu lectures)

### 6.1 Hodge theory in the conformal method (summary)

Hodge theory is frequently used in conformal decompositions of tensor fields on manifolds, notably in the study of constraint equations in general relativity where one decomposes vector fields into divergence-free and gradient parts using Hodge decomposition on forms (or vector Laplacians). This decomposition underlies the *conformal method*, where initial data for Einstein equations is constructed by solving elliptic PDEs whose solvability depends on the Hodge-type decompositions and the Fredholm properties of the associated elliptic operators.

### 6.2 Corvino's three lectures (brief summary)

Corvino's lectures (summarized in the assignment prompt) discuss gluing and deformation techniques in geometric analysis that exploit elliptic theory (Hodge theory and the conformal method) to glue asymptotically flat metrics, construct localized deformations of initial data, and handle constraints using functional-analytic tools (Fredholm theory, weighted Sobolev spaces). Hodge theoretic decompositions help decouple equations into solvable elliptic systems.

## 6.3 Potential future problems

One direction mentioned is the conformal method for conformal gravity: adapting conformal decomposition and constraint-solving techniques to higher-derivative conformal theories requires extended elliptic theory for higher-order operators, careful study of functional spaces (weighted Sobolev spaces), and refined parametrix constructions. Hodge-theoretic insights (harmonic obstructions, index considerations) remain central in understanding solvability and moduli.

Rudin Real and Complex Analysis
As Reference

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$D(\Omega)$ = Space of test function space (of compactly supported smooth functions) are dense in $W^{k,p}$

$W^{k,p}$ : $1 \le p \le \infty$, $k$ = non-negative integer.

If we define a measure

$$d\mu_k(y) = (1+|y|^2)^k \, dm_n(y)$$

So, $f \in W^{k,p}$ means $\int |f|^p \, d\mu_k(y) < \infty$

Now as $k \ge 0$, so $(1+|y|^2)^k \ge 1$, so

$$\int |f|^p \, dm_n \le \int |f|^p (1+|y|^2)^k \, dm_n$$

so, $f \in L^p(\mathbb{R}^n)$

Norm on $W^{k,p}$ is same as that of $L^p$ so

we will prove that $D(\Omega)$ is dense in

$L^p(\mathbb{R}^n)$ hence will be dense in $W^{k,p}$.

Let $X = \mathbb{R}^n$ (the argument is the same for any manifold locally like $\mathbb{R}^n$). Let

$$\mathcal{D} = C_c^\infty(\mathbb{R}^n)$$

be the space of smooth compactly supported functions, and let

$$C_c(\mathbb{R}^n)$$

be the space of continuous compactly supported functions, both equipped with the uniform (sup) norm $\|g\|_\infty = \sup_{x\in\mathbb{R}^n} |g(x)|$. I will show $\overline{\mathcal{D}}^{\|\cdot\|_\infty} = C_c(\mathbb{R}^n)$; i.e. every $f \in C_c(\mathbb{R}^n)$ can be uniformly approximated by elements of $\mathcal{D}$.

**Step 1 — choose a standard mollifier.**
Let $\rho \in C_c^\infty(\mathbb{R}^n)$ satisfy $\rho \geq 0$, $\operatorname{supp}\rho \subset B(0,1)$ and $\int_{\mathbb{R}^n} \rho = 1$. Set

$$\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon) \qquad (\varepsilon > 0).$$

Each $\rho_\varepsilon$ is in $C_c^\infty$ with $\operatorname{supp}\rho_\varepsilon \subset B(0,\varepsilon)$ and $\int \rho_\varepsilon = 1$.

As from 6.8(b) and 6.11 (3), we can find such a $\rho$ with $K=B(0,1)\subset R^n$.

**Step 2 — convolution and support.**
For $f \in C_c(\mathbb{R}^n)$ define the convolution

So $\Lambda_f$ will be in $D'(R^n)$ as f is locally integrable

$$f_\varepsilon := f * \rho_\varepsilon.$$

See the convolution part Rudin in pg 170, 6.30(b)

Because $f$ is continuous and compactly supported, $f_\varepsilon$ is smooth. Moreover

As f and $\rho_\varepsilon$ are in multiplied form in $f_\varepsilon$

$$\operatorname{supp}(f_\varepsilon) \subset \operatorname{supp}(f) + \operatorname{supp}(\rho_\varepsilon),$$

so $\operatorname{supp}(f_\varepsilon)$ is compact (indeed a small enlargement of $\operatorname{supp} f$). Hence $f_\varepsilon \in C_c^\infty(\mathbb{R}^n) = \mathcal{D}$.

**Step 3 — uniform convergence.**
Since $f$ has compact support it is uniformly continuous. Fix $\eta > 0$. By uniform continuity there exists $\delta > 0$ such that $|f(x-y) - f(x)| < \eta$ whenever $|y| < \delta$. Choose $\varepsilon$ with $0 < \varepsilon < \delta$. Then for every $x$,

$$\begin{aligned} |f_\varepsilon(x) - f(x)| &= \left| \int_{\mathbb{R}^n} \big(f(x-y) - f(x)\big)\rho_\varepsilon(y)\,dy \right| \\ &\leq \int_{\mathbb{R}^n} |f(x-y) - f(x)|\,\rho_\varepsilon(y)\,dy. \end{aligned}$$

But $\rho_\varepsilon$ is supported in $B(0,\varepsilon)$, so for all $y$ in the integral $|y| < \varepsilon < \delta$, hence $|f(x-y) - f(x)| < \eta$. Therefore

$$|f_\varepsilon(x) - f(x)| \leq \int \eta\,\rho_\varepsilon(y)\,dy = \eta.$$

This is true for every $x$, so $\|f_\varepsilon - f\|_\infty \leq \eta$. Since $\eta > 0$ was arbitrary, $\|f_\varepsilon - f\|_\infty \to 0$ as $\varepsilon \to 0$.

**Conclusion.**
For each $\varepsilon > 0$ small, $f_\varepsilon \in \mathcal{D}$ and $f_\varepsilon \to f$ uniformly. Thus $f$ belongs to the uniform closure of $\mathcal{D}$. Because $f \in C_c(\mathbb{R}^n)$ was arbitrary, $\overline{\mathcal{D}}^{\|\cdot\|_\infty} = C_c(\mathbb{R}^n)$.

Let $f \in L^1(\mathbb{R}^n)$ and let $\varepsilon > 0$. By Theorem 3.14 in *Rudin, Real and Complex Analysis*, $C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, so there exists

$$f_c \in C_c(\mathbb{R}^n) \quad \text{such that} \quad \|f - f_c\|_1 < \frac{\varepsilon}{2}.$$

From page 150 of Rudin, we know that

$$C_c(\mathbb{R}^n) \subset \overline{D(\mathbb{R}^n)}$$

(where the closure is taken in the supremum norm). Thus there exists $f_D \in D(\mathbb{R}^n)$ such that

$$\|f_c - f_D\|_\infty < \frac{\varepsilon/2}{\mu(K+B)},$$

where $K = \operatorname{supp}(f_c)$ and $B$ is the closed unit ball (see the page following p. 150). Since $f_c - f_D \in C_c(\mathbb{R}^n)$, let $K' = \operatorname{supp}(f_c - f_D)$; then $K' \subset K + B$.

Now apply Theorem 3.8 of Rudin with $p = \infty$ and $q = 1$. There exists a compactly supported function $g \in L^1(\mathbb{R}^n)$ such that:
- $g = 1$ on $K'$, - $g = 0$ outside $K'$.

Thus

$$(f_c - f_D)g = f_c - f_D.$$

Hence,

$$\|f_c - f_D\|_1 = \|(f_c - f_D)g\|_1 \leq \|f_c - f_D\|_\infty \, \|g\|_1 < \frac{\varepsilon/2}{\mu(K+B)} \cdot \mu(K') < \frac{\varepsilon}{2} \frac{\mu(K')}{\mu(K+B)} \leq \frac{\varepsilon}{2}.$$

Finally,

$$\|f - f_D\|_1 \leq \|f - f_c\|_1 + \|f_c - f_D\|_1 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $f_D \in D(\mathbb{R}^n)$ approximates $f$ in the $L^1$ norm arbitrarily well.

We now show that $T_n$ is dense in $L^2(\mathbb{R}^n)$. The proof is very similar to the case of $L^1(\mathbb{R}^n)$.

Let $\varepsilon > 0$. Choose $f_c \in C_c(\mathbb{R}^n)$ such that

$$\|f - f_c\|_2 < \frac{\varepsilon}{2}.$$

From the discussion following page 150 of Rudin, each $f_D$ has the form

$$f_D = f_c * \rho_\varepsilon,$$

which is compactly supported, with

$$\operatorname{supp}(f_D) \subset K + B,$$

where $K = \operatorname{supp}(f_c)$ and $B$ is the closed unit ball. Let

$$f_m = \sup_{x \in \mathbb{R}^n} |f_c(x)| < \infty.$$

Since $\rho_\varepsilon \leq 1$, we have

$$\|f_D\|_1 \leq f_m \, \mu(K + B) \, \mu(K + B) = f_m \, (\mu(K + B))^2,$$

and similarly

$$\|f_c\|_1 \leq f_m \, \mu(K + B).$$

Now take $f_D$ satisfying

$$\|f_c - f_D\|_\infty < \frac{\varepsilon/2}{f_m \, \mu(K + B) \, (1 + \mu(K + B))}.$$

Apply Theorem 3.8 of Rudin with $f = g = f_c - f_D$:

$$\|fg\|_1 = \|f_c - f_D\|_2^2 \leq \|f_c - f_D\|_\infty \, \|f_c - f_D\|_1.$$

Since

$$\|f_c - f_D\|_1 \leq \|f_c\|_1 + \|f_D\|_1 \leq f_m \, \mu(K + B) \, (1 + \mu(K + B)),$$

we obtain

$$\|f_c - f_D\|_2^2 < \frac{\varepsilon}{2}.$$

Thus

$$\|f - f_D\|_2 \leq \|f - f_c\|_2 + \|f_c - f_D\|_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $T_n$ is dense in $L^2(\mathbb{R}^n)$.

Rudin Real and Complex :

Theorem: 3.8: $q, p$ conjugate, $1 \leq p \leq \infty$ $f \in L^p(\mu)$

$g \in L^q(\mu)$, $fg \in L^1(\mu)$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

so, we can similarly take $g = |f_\varepsilon - f_D|$, $f = |f_c - f_D|^2$

$$\| |f_c - f_D|^3 \|_1 = \|f_c - f_D\|_3 \leq \|f_\varepsilon - f_D\|_2 \, \|f_c - f_D\|_2$$

so in this way, we can prove $D(\Omega)$ is dense in $L^p(\mathbb{R}^n)$

Hence $\mathcal{L}_n$ is dense in $L^p(\mathbb{R}^n)$, $D(\Omega)$ is $W^{k,p}$.

We are to prove that if $f \in L^1_{\text{loc}}(\Omega)$ satisfies $\int_\Omega f\varphi = 0$ for all $\varphi \in C_0^\infty(\Omega)$, then $f = 0$ a.e. on $\Omega$.

Let $K$ be any compact subset of $\Omega$, and take $\psi \in C_0^\infty(\Omega)$ such that $\psi = 1$ on $K$. Define a function $f_\psi$ on $\mathbb{R}^n$ as

$$f_\psi(x) = \begin{cases} 0 & x \notin \Omega, \\ f(x)\psi(x) & x \in \Omega. \end{cases}$$

which extends $f$ canonically to the whole $\mathbf{R}^n$. Then $f_\psi \in L^1(\mathbf{R}^n)$. Pick a mollifier $\varphi \in C_0^\infty(\mathbf{R}^n)$. Note that $\varphi_\epsilon * f_\psi \to f_\psi$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ as $\epsilon \to 0$. We have

$$\varphi_\epsilon * f_\psi(x) = \int f(y)\psi(y)\varphi_\epsilon(x-y)dy,$$

and note that for a fixed $x$, the map $y \mapsto \psi(y)\varphi_\epsilon(x-y) \in C_0^\infty(\Omega)$, so $\varphi_\epsilon * f_\psi = 0$ by hypothesis. As $\epsilon \to 0$, we have $\varphi_\epsilon * f_\psi \to f_\psi$ and thus $f_\psi = 0$ a.e. on $\Omega$

Hence $f = 0$ a.e. on $K$. Since $K$ is arbitrary compact subset of $\Omega$, we have $f = 0$ a.e. on $\Omega$.

See Plancharel's theorem and the pages below that for similar convergence. It's due to Fubini's theorem.

Let $R^n = \cup K_i$ where $K_i$ are compact and $K_i \subset K_{i+1}$,(It's possible-See 1.44-Rudin Functional analysis) Let $E_i = \{x \mid f(x) \neq 0, x \in K_i\}$, so $\mu(E_i)=0$ for each i. Now we need to find the measure of $E = \cup_i E_i$. But as $E_i \subset E_{i+1}$, we have $\mu(E_i) \to \mu(E)$(See rudin real and complex analysis-1.19(d)) so $0 \to 0$. So $\mu(E)=0$.

Given $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, we want to find a sequence $\{\phi_k\} \subset \mathscr{S}$ (Schwartz functions) such that:

1. $\|\phi_k - f\|_1 \to 0$
2. $\|\phi_k - f\|_2 \to 0$

We will actually prove a stronger statement: we can find such a sequence in $C_c^\infty(\mathbb{R}^n)$ (smooth functions with compact support), which is a subset of the Schwartz space.

**Step 1: Approximation by Compact Support (Truncation)**

First, we approximate $f$ with a function that is zero outside of a large ball.

Let $B_k$ be the ball of radius $k$ centered at the origin. Define the truncated function $g_k$ as:

$$g_k(x) = f(x) \cdot 1_{B_k}(x) = \begin{cases} f(x) & \text{if } |x| \le k \\ 0 & \text{if } |x| > k \end{cases}$$

**Convergence in $L^1$:**

$$\int |g_k - f| = \int_{|x|>k} |f|$$

Since $f \in L^1$, the "tail" integral goes to 0 as $k \to \infty$ (by the Dominated Convergence Theorem, dominated by $|f|$). Thus, $\|g_k - f\|_1 \to 0$.

The measure (See 1.29 in Rudin real and complex analysis) $F(A_k)=\int_{|x|\le k}|f|^2$ is monotonous for $A_k=\{x \mid |x|\le k\}$ and $R^n=\bigcup_{k=1 \text{ to } \infty}A_k$ with $A_k \subset A_{k+1}$. So we can use 1.19(d)(Rudin real and complex analysis) to see that $F(A_k)\to\int_{R^n}||f||^2=C$. So now Let $F'_k=\int_{|x|>k}|f|^2$ and $F'_k+F(A_k)= C$ so with $k \to \infty$,on the lhs, $F(A_k)\to C$ so $F'_k$ must tend to 0

**Convergence in $L^2$:**

$$\int |g_k - f|^2 = \int_{|x|>k} |f|^2$$

Since $f \in L^2$, $|f|^2$ is integrable, so its tail integral also goes to 0. Thus, $\|g_k - f\|_2 \to 0$.

**Result:** The sequence $\{g_k\}$ approximates $f$ in both norms, but the functions $g_k$ are not smooth (they have sharp cutoffs).

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**Step 2: Approximation by Smoothing (Mollification)**

Now we smooth out the sharp edges of $g_k$.

Let $\eta$ be a standard **mollifier**: a smooth function in $C_c^\infty$ that is non-negative, supported in the unit ball, and satisfies $\int \eta = 1$. Define the scaled mollifier $\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon)$.

We define our candidate sequence by convolving the truncated function $g_k$ with the mollifier (choosing $\epsilon = 1/k$ for convenience):

$$\phi_k = g_k * \eta_{1/k}$$

**Why is $\phi_k$ in the Schwartz space?**

1. **Smoothness:** Convolution with a $C_c^\infty$ function produces a $C^\infty$ function.
2. **Compact Support:** $g_k$ has support in $|x| \le k$, and $\eta_{1/k}$ has support in $|x| \le 1/k$. Therefore, $\phi_k$ is supported in $|x| \le k + 1/k$.
3. Any smooth function with compact support is automatically a Schwartz function ($C_c^\infty \subset \mathscr{S}$ ).

**Step 3: Proving Simultaneous Convergence**

We use the triangle inequality to split the error into two parts: the error from truncating ($f \to g_k$) and the error from smoothing ($g_k \to \phi_k$).

For any norm $p$ (where $p = 1$ or $p = 2$):

$$||\phi_k - f||_p \le \underbrace{||\phi_k - g_k||_p}_{\text{Smoothing Error}} + \underbrace{||g_k - f||_p}_{\text{Truncation Error}}$$

1. **Truncation Error:** We already proved in Step 1 that $||g_k - f||_p \to 0$ as $k \to \infty$ for both $p = 1$ and $p = 2$.
2. **Smoothing Error:** A standard property of approximate identities (mollifiers) states that if $g \in L^p$, then $||g * \eta_\epsilon - g||_p \to 0$ as $\epsilon \to 0$.
   - Since $g_k \in L^1 \cap L^2$, the convolution converges to $g_k$ in both norms.

See the two pages above -There g was actually in $C_c(R^n)$ but here g isn't. So new technology is required. So $g_k$ is still locally integrable so $\Lambda_{g_k}$ will be a distribution(Lebesgue) and so $\Phi_k = g_k * \rho_{1/k}$ ($\rho \equiv \eta$ here) will still be in $D(R^n) \subset C^\infty(R^n)$ with support = Ball of radius k+1/k. We will start from considering $k \ge 1$. So now for $||\Phi_k - g_k||_p = \int_{R^n} |(\Phi_k(x) - g_k(x))|^p dx = \int_{R^n} | \int_{R^n} \rho_{1/k}(y) g_k(x-y) dy - \int_{R^n} \rho_{1/k}(y) g_k(x) dy |^p dx \le \int_{R^n} (\int_\Omega |\rho_{1/k}(y)| \, |(g_k(x-y) - g_k(x))| dy)^p dx$, here $\Omega$=Ball of radius 1 as $\rho_{1/k}$ has always support inside $\Omega$ for $k \ge 1$. Now we will use Jensen's inequality(See 3.3-Rudin real and complex analysis) with the fact that $\varphi = t^p$ is a convex function for $0 < t < \infty$-See last of pg 64 there: So $||\Phi_k - g_k||_p \le \int_{R^n} \int_\Omega |\rho_{1/k}(y)|^p \, |g_k(x-y) - g_k(x)|^p \, dy \, dx$. Now using Fubini's theorem(See 8.8 in Rudin real and .....) we have $||\Phi_k - g_k||_p \le \int_\Omega \rho_{1/k}(y)^p \, (\int_{R^n} |g_k(x-y) - g_k(x)|^p dx) dy$,now for p=1 or 2, we know $g_k$ will be $L^p$ as f is $L^p$. So $|g_k(x-y) - g_k(x)| \le |g_k(x-y)| + |g_k(x)| \le |f(x-y)| + |f(x)| = L^p$ function so $\int_{R^n} |g_k(x-y) - g_k(x)|^p \le \int_{R^n} (|f(x-y)| + |f(x)|)^p dx = R < \infty$. So $||\Phi_k - g_k||_p \le \mu\{x: | \, |x| < 1/k\} * R < R'/k^n$.So $||\Phi_k - g_k||_p$ can be brought as close to 0 as possible with $k \to \infty$. So done

**Technical Detail:** Strictly speaking, for a fixed $k$, we choose an $\epsilon_k$ small enough such th
$||\phi_k - g_k||_1 < 1/k$ and $||\phi_k - g_k||_2 < 1/k$. (We can do this because convergence ho
fixed function).

**Conclusion**

As $k \to \infty$:

- The truncation error vanishes because $f$ decays at infinity.
- The smoothing error vanishes because the mollifier shrinks to a Dirac delta.

Thus, $||\phi_k - f||_1 \to 0$ and $||\phi_k - f||_2 \to 0$ simultaneously. Q.E.D.

**DEFINITION. The Hölder space $C^{k,\gamma}(U)$**

Consists of all functions $u \in C^k(\bar{U})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\bar{U})} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})}$$

is **finite**.

> **Note:** $\alpha$ is a vector like $(1, 3, 5)$ so that $D^\alpha \equiv \frac{\partial^{|\alpha|}}{\partial x_1^1 \partial x_2^3 \partial x_3^5}$. So it means the sum of all partial derivatives possible.

# Completeness of Hölder Spaces

**Reference:** See page 90 of Marsden's *Manifolds* (Textbox at the very end).

*Proof.* It is established that $C(U \subset \mathbb{R}^n, E = \mathbb{R})$ is a Banach space with respect to the standard $C^k$ norm:

$$\|u\|_{C^k} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})}$$

(Note: In the referenced textbox, the proof is general for $U \subset \mathbb{R}^n$, not merely for $I \subset \mathbb{R}$).

Consequently, any Cauchy sequence $\{u_n\}$ in the Hölder space $C^{k,\gamma}(\bar{U})$ will also be a Cauchy sequence in $C^k(\bar{U})$ with respect to the $C^k$ norm. Hence, it converges to some limit $u \in C^k(\mathbb{R}^n, \mathbb{R})$.

Now, we need to show that for all possible multi-indices with $|\alpha| = k$, the function $f = D^\alpha u$ will also have a finite $\gamma$-th Hölder seminorm, and that $f_n = D^\alpha u_n$ will converge to $f = D^\alpha u$ in that norm.

**Finiteness of the Limit's Seminorm** Note that $f_n \to f$ in the sup norm (hence uniformly). Thus, $|f(x) - f_n(x)| < \epsilon$ for all $x$, for some $n > N_\epsilon$.

Since $\{f_n\}$ is Cauchy in the Hölder norm, $[f_n - f_m]_\gamma < \epsilon$ for some $n, m > N'_\epsilon$. Observe the reverse triangle inequality for the seminorm:

$$[f_n - f_m]_\gamma = \sup_{x \neq y} \frac{|(f_n(x) - f_m(x)) - (f_n(y) - f_m(y))|}{|x - y|^\gamma} \geq |[f_n]_\gamma - [f_m]_\gamma|$$

Therefore, the sequence of real numbers $\{[f_n]_\gamma\}$ is Cauchy in $\mathbb{R}$, hence it converges, and hence it is finite. Thus $\{[f_n]_\gamma\}$ is bounded; let $M$ be a bound such that $[f_n]_\gamma \leq M$ for all $n$.

Now consider the difference quotient for the limit $f$:

$$\begin{aligned}
\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma} &= \sup_{x \neq y} \frac{|f(x) - f_m(x) + f_m(x) - (f(y) - f_m(y)) - f_m(y)|}{|x - y|^\gamma} \\
&\leq \sup_{x \neq y} \frac{|(f(x) - f_m(x)) - (f(y) - f_m(y))|}{|x - y|^\gamma} + [f_m]_\gamma \\
&\leq \sup_{x \neq y} \frac{|f(x) - f_m(x)|}{|x - y|^\gamma} + \sup_{x \neq y} \frac{|f(y) - f_m(y)|}{|x - y|^\gamma} + [f_m]_\gamma
\end{aligned}$$

For any fixed separation $|x - y|^\gamma = K$, we can always find $m > N_K$ such that $|f(x) - f_m(x)| < 1$ and $|f(y) - f_m(y)| < 1$. Thus:

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\gamma} < M + 1 + 1 = M + 2$$

So $[f]_\gamma$ is finite.

**Convergence in Seminorm** Finally, to show that $f_m \to f$ in the Hölder Seminorm sense (that is, $[f - f_m]_\gamma \to 0$), see the referenced page below. Thus, all derivatives $D^\alpha u$ are accounted for, completing the proof.

□

We already know $v$ has finite Hölder seminorm $[v]$ and if $v_n$ is Cauchy in the Hölder seminorm, so will be $p_n = v_n - v$, so using norm inequality:

$$\big|[p_n] - [p_m]\big| = \big|[v_n - v] - [v_m - v]\big| \le [(v_n - v) - (v_m - v)] < \varepsilon$$

for all $m, n > N$, for some $N$. So $[p_n] = [v_n - v]$ is a Cauchy sequence in $\mathbb{R}$, hence it will converge as $\mathbb{R}$ is complete.

Let's assume it converges to some nonzero positive value $r$. Let $\varepsilon$ be a positive number such that $r - \varepsilon > 0$.

We know:

$$\begin{aligned}[v_n - v] &= \sup_{x \neq y} \frac{|v_n(x) - v(x) - (v_n(y) - v(y))|}{|x - y|^\gamma} \\ &\le \sup_{x \neq y} \frac{|v_n(x) - v(x)|}{|x - y|^\gamma} + \sup_{x \neq y} \frac{|v_n(y) - v(y)|}{|x - y|^\gamma}. \end{aligned} \qquad (1)$$

Now as $[v_n - v]$ converges to $r$ , for the ε, there is a N such that $[v_n - v]$ > r - ε for all n > N. So for each n>N, there is at least one $(x, y)$ with $x \neq y$ such that:

$$\frac{|v_n(x) - v(x) - (v_n(y) - v(y))|}{|x - y|^\gamma} > r - \varepsilon. \qquad (2)$$

Now again: As $v_m$ converges in the sup norm hence uniformly to $v$, we can choose another $N_{xy}'$ (which is (x,y) dependent) such that for all $m > N_{xy}'$:

$$\frac{|v_m(x) - v(x)|}{|x - y|^\gamma} < \frac{r - \varepsilon}{4}$$

and

$$\frac{|v_m(y) - v(y)|}{|x - y|^\gamma} < \frac{r - \varepsilon}{4},$$

so that using equation (1):

$$\frac{|v_m(x) - v(x) - (v_m(y) - v(y))|}{|x - y|^\gamma} < \frac{r - \varepsilon}{2}. \qquad (3)$$

for all $m > N_{xy}'$ and for the (x,y), x≠y.

Now $v_n$ being cauchy, there is a M such that for n,m>M, $[v_n - v_m]$<(r-ε)/2 so for all n , m > M:

$$\begin{aligned}\frac{|v_n(x) - v(x) - (v_n(y) - v(y))|}{|x - y|^\gamma} &= \frac{|v_n(x) - v_m(x) - (v_n(y) - v_m(y)) + v_m(x) - v(x) - (v_m(y) - v(y))|}{|x - y|^\gamma} \\ &\le \frac{|v_n(x) - v_m(x) - (v_n(y) - v_m(y))|}{|x - y|^\gamma} + \frac{|v_m(x) - v(x) - (v_m(y) - v(y))|}{|x - y|^\gamma} \\ &\le [v_n - v_m] + \frac{|v_m(x) - v(x) - (v_m(y) - v(y))|}{|x - y|^\gamma} \\ &\le (r - \varepsilon)/2 + \frac{|v_m(x) - v(x) - (v_m(y) - v(y))|}{|x - y|^\gamma} \end{aligned} \qquad (4)$$

So for any n>max(M,N), for any (x,y),x≠y, we can take m>max(M,$N_{xy}$) so that,using equation (3) and (4), we will have:

$$\frac{|v_n(x) - v(x) - (v_n(y) - v(y))|}{|x - y|^\gamma} < r - \varepsilon. \qquad (5)$$

for all n>max(M,N) for each (x,y),x≠y.

But clearly equation (2) and (5) pose a contradiction. Because for any n>max(M,N), we should be able to find a (x,y),x≠y such that (2) is true. Therefore, $[v_n - v]$ must converge to $r = 0$. Hence $v_n$ converges to $v$ in the Hölder seminorm sense too.