ICTP PWF: Bangladesh Internship Program

Project : Properties of Scattering Amplitudes

Project Report – October 2025

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1 Introduction

In quantum field theory, loop integrals appearing in Feynman diagrams often become difficult to evaluate because they involve integrations over internal, undetermined momenta. Such integrals frequently give rise to divergences, particularly when the propagator denominators develop poles. Once these denominators are made explicit, the Feynman amplitude becomes a complex function of the external momenta, making its analytic properties an important object of study.

The analytic structure of an amplitude reveals where it behaves smoothly and where it encounters singularities. These singular points signal transitions from internal momenta to physically allowed ones. Landau's formulation provides a geometric framework to understand where such singularities appear, while the Landau equations describe the precise mathematical conditions—combining propagator denominators and momentum-conservation constraints—that lead to a so-called *pinch singularity*.

This report explores these ideas with reference to the work *"The Analytic Properties of Perturbation Theory–I"* by J. C. Polkinghorne and G. R. Screaton 1. Their analysis employs contour deformation in the complex momentum plane to identify all possible singularities in Feynman amplitudes. They show that singular behavior arises either when the Feynman parameters reach boundary values or when poles in the integrand pinch the contour, both of which are described precisely by the Landau equations. Understanding these conditions clarifies the analytic structure underlying perturbative quantum field theory.

2 Analytic Structure of Feynman Integrals

Building upon the above discussion, we now focus on understanding the *analytic properties of Feynman integrals* themselves, but without performing the full, explicit computation of the integrals. Instead, our analysis is carried out at the *integrand level*, allowing us to identify and study the singularities that arise directly from the internal structure of the integrand.

It is essential to understand the analytic structure of the amplitude, since this determines the appropriate *integration contour* in the complex plane. Over the years, the analytic properties of Feynman integrals have been investigated in various contexts. Earlier studies include analysis of the *Landau equations* and *physical-region singularities*, the *unitarity relations* involving two- and three-particle intermediate states, as well as detailed examinations of *specific physical processes*, such as pion–pion ($\pi\pi$) scattering. Additional research has explored *spectral representations* and the *Mandelstam representation* within perturbative field theory. Together, these studies highlight the central role that analytic properties play in understanding the fundamental behavior of scattering amplitudes in high-energy physics.

Typically, once the analytic computation of a Feynman integral is completed, its analytic structure and singularities can be directly extracted from the result. However, our focus here is to explore *methods that allow this analysis without performing the full integration*, relying instead on the geometric and algebraic properties of the integrand itself.

Formally, the analytic structure of Feynman integrals is most often studied using two main tools:

1. **Landau Equations:** These provide the conditions under which singularities arise in Feynman integrals by identifying when propagator denominators and momentum-conservation constraints simultaneously lead to a *pinch singularity.*

2. **Cutkosky Rules:** Cutkosky Rules for Feynman diagrams are required for Unitarity of the S-matrix i.e. $S^\dagger S = I$ .These rules are used to determine the *discontinuities* of amplitudes across branch cuts, thus connecting the analytic behavior of an amplitude with physical intermediate states that satisfy on-shell conditions.

In this report, we employ one methods—*Landau analysis*—to investigate the singularities and analytic structure of Feynman integrals without explicitly performing their integration. This approach allows us to understand how real physical processes emerge from the analytic properties of quantum amplitudes.

It is worth mentioning that the ideas of analyticity and continuation appear in many other areas of quantum field theory as well. For example, *crossing symmetry* relates scattering amplitudes of different processes through analytic continuation by interchanging incoming and outgoing particles. Similarly, techniques such as *dimensional regularization* make use of analytic continuation in the number of spacetime dimensions to control divergences in loop integrals.

2.1 Singularities and Landau Equations

In quantum field theory, loop integrals naturally appear in Feynman diagrams and are often challenging to evaluate due to integration over undetermined internal momenta. A general n-propagator loop integral in D dimensions can be written as

$$f(p) = \int \frac{d^D k}{(2\pi)^D} \prod_{i=1}^{n} \frac{1}{q_i^2 - m_i^2 + i\epsilon}, \tag{1}$$

where q_i are linear combinations of loop momenta k and external momenta p, and m_i are the masses of the internal lines.

These integrals often lead to divergences, particularly when the propagator denominators vanish. When the denominators are made explicit, the amplitude becomes a complex function of the external momenta $\{p_s\}$, making its analytic properties a topic of fundamental interest. Analytic continuation of $f(p)$ is possible except at certain special points, which can be classified as:

- **End-point singularities:** a singularity crosses the integration contour and cannot be avoided by bending the contour.

- **Pinch singularities:** two singularities approach the contour from opposite sides, effectively *pinching* it.

- **Singularities at infinity:** one or more singularities recede to infinity for specific values of ζ.

To simplify the evaluation of loop integrals and to study their singularities, one introduces **Feynman parameters** α_i, which allow all propagators to be combined into a single denominator. The integral then becomes

$$f(\{p_s\}) = \int \prod_i d\alpha_i \, \delta\Big(\sum_i \alpha_i - 1\Big) \int \prod_r d^4 k_r \, \frac{F(\alpha_i, k_r, p_s)}{D(\alpha_i, k_r, p_s)} \tag{2}$$

Here, $D(\alpha_i, k_r, p_s) = \sum_j \alpha_j \left[l_j^2(\{p\}, \{k\}) - m_j^2 \right] + i\epsilon$, and $F(\alpha_i, k_r, p_s)$ represents constant and numerator factors that do not affect the arguments presented below. The parameter α_j is the Feynman parameter of the jth line, and $l_j(p, k)$ is the momentum of that line, a linear function of the loop momenta $\{k_r\}$ and external momenta $\{p_s\}$.

Our goal is to determine the positions of all poles and branch points of $f(\{p_s\})$ as functions of $\{p_s\}$. These singularities arise from zeros of the denominator $D(\alpha_i, k_r, p_s)$. In the absence of such zeros, the integrand in Eq. (2) is analytic throughout the integration region, and hence $f(\{p_s\})$ is an analytic function of the external momenta $\{p_s\}$.

The Landau equations arise from analyzing the conditions under which both the denominator of a Feynman integral and its derivatives with respect to the loop momenta vanish simultaneously. This situation corresponds to *pinch singularity*, where the integration contour in complex momentum space cannot be deformed away from the singularity.

When the denominator of the Feynman integral vanishes, the singularity appears. For a loop with momenta k_r, and the condition for a singularity can be expressed as

$$\sum_{\text{lines } i} \alpha_i \left[l_i^2(p, k) - m_i^2 \right] + i\epsilon = 0, \tag{3}$$

$$l^\mu(p, k) = \sum_r \eta_{ir} k_r^\mu + \sum_s \hat{\eta}_{is} p_s^\mu, \tag{4}$$

where η_{ir} and $\hat{\eta_{is}}$ are incidence matrices corresponding to loop and external momenta, respectively. The contour for the loop momentum k_j^ν can become trapped only if two solutions coincide, which occurs when:

$$0 = \frac{\partial}{\partial k_j^\nu} \left\{ \sum_i \alpha_i \left[l_i^2(p, k) - m_i^2 \right] + i\epsilon \right\} = 2 \sum_i \eta_{ij} (\alpha_i l_i^\mu). \tag{5}$$

Now, let us introduce $\xi = \alpha_i$, a Feynman parameter. The function $D(\alpha_i, k_r, p_j)$ is independent of α_i when $l_i^2 = m_i^2$. Since D is linear in each α_i, the contour can be trapped only at its endpoint $\alpha_i = 0$.

These conditions are conveniently summarized as the *Landau equations*. A surface S is a pinch surface only if the following conditions hold for each point (k^μ, α_i) on S:

- For lines with $l_i^2 = m_i^2$, we have, for every loop j that includes line i,

$$\sum_i \eta_{ij} \alpha_i l_i^\mu = 0. \tag{6}$$

- For lines with $l_i^2 \neq m_i^2$, we have

$$\alpha_i = 0. \tag{7}$$

Equations (6) and (7) together form the complete set of Landau conditions for identifying singularities in Feynman diagram amplitudes.

2.2 Derivation of Landau equations for non-local vertices

We start from the generic n-propagator Feynman loop integral:

$$f(p) = \int \frac{d^D k}{(2\pi)^D} \prod_{i=1}^{n} \frac{1}{[(q_i^2 - m_i^2 + i\epsilon)]},$$

where $q_i = k + p_i$ are the internal momenta.

Using the Schwinger parametrization,

$$\frac{1}{A_i} = \int_0^\infty ds_i \, e^{-is_i A_i},$$

we can write

$$f(p) = \int \prod_{a=1}^{m} \frac{d^D k_a}{(2\pi)^D} \int_0^\infty \prod_{i=1}^{n} d\beta_i \, \exp\left[\sum_{i=1}^{n} \beta_i (q_i^2 - m_i^2 + i\epsilon)\right]$$

$$= \int_0^\infty \prod_{i=1}^{n} d\beta_i \, e^{\sum_i \beta_i (m_i^2 - i\epsilon)} \int \prod_{a=1}^{m} d^D k_a \exp\left[-i \sum_i \beta_i (k_i + p_i)^2\right],$$

where using

$$q_i = \sum_r C_{ir} k_r + \sum_s Q_{is} p_i;$$

But C_{ir} and Q_{is} are just constants, so we are not taking them into account. In the limit $\epsilon \to 0$ the integral becomes

$$= \int_0^\infty \prod_{i=1}^{n} d\beta_i \, e^{\sum_i \beta_i m_i^2} \int \prod_{a=1}^{m} d^D k_a \exp\left[-i \sum_i \beta_i (k_i + p_i)^2\right].$$

$$= \int_0^\infty \prod_{i=1}^{n} d\beta_i \, e^{-\sum_i \beta_i (p_i^2 - m_i^2)} \int \prod_{a=1}^{m} d^D k_a \exp\left[\sum_\beta i(k_i^2 + 2k.p)\right].$$

Changing variable k and making it Gaussian, we get the form as

$$= \int_0^\infty \prod_{i=1}^{n} d\beta_i \, e^{-\nu} \int \prod_{a=1}^{m} d^D \tilde{k_a} \exp\left[(k_i^2 \phi(\beta_i)\right].$$

$$= \int_0^\infty \prod_{i=1}^{n} d\beta_i \frac{\pi^{mD/2}}{\prod_j [\phi(\beta_j)]^{mD/2}} e^{-\nu(\alpha_i, p_i)}$$

Now changing the variable (Feynmann parametrization)

$$\beta_i = \beta \, \alpha_i, \qquad \sum_i \beta_i = \beta$$

We are changing the variable from

$$(\beta_1, \beta_2, \ldots, \beta_n) \;\longleftrightarrow\; \big(\beta, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}\big),$$

$$\text{where } \alpha_n = 1 - \sum_{i=1}^{n-1} \alpha_i.$$

$$\mathrm{d}\beta_1 \wedge \mathrm{d}\beta_2 \wedge \cdots \wedge \mathrm{d}\beta_n = \det\left(\frac{\partial(\beta_1, \ldots, \beta_n)}{\partial(\beta, \alpha_1, \ldots, \alpha_{n-1})} \right) \mathrm{d}\beta \wedge \mathrm{d}\alpha_1 \wedge \cdots \wedge \mathrm{d}\alpha_{n-1}.$$

Now,

$$\frac{\partial \beta_i}{\partial \beta} = \alpha_i, \qquad \frac{\partial \beta_i}{\partial \alpha_j} = \begin{cases} \beta & (i = j) \\ -\beta & (i = n) \\ 0 & (\text{otherwise}). \end{cases}$$

$$J \;=\; \begin{pmatrix} \alpha_1 & \beta & 0 & \cdots & 0 \\ \alpha_2 & 0 & \beta & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1} & 0 & 0 & \cdots & \beta \\ \alpha_n & -\beta & -\beta & \cdots & -\beta \end{pmatrix}.$$

$$\det J = \beta^{\,n-1} \det \begin{pmatrix} \alpha_1 & 1 & 0 & \cdots & 0 \\ \alpha_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-1} & 0 & 0 & \cdots & 1 \\ \alpha_n & -1 & -1 & \cdots & -1 \end{pmatrix}.$$

Jacobian absolute value-

$$\det\left| \frac{\partial(\beta_1, \ldots, \beta_n)}{\partial(\beta, \alpha_1, \ldots, \alpha_{n-1})} \right| = \beta^{\,n-1}.$$

So,

$$\prod_{i=1}^{n} \mathrm{d}\beta_i = \det\left| \frac{\partial(\beta_1, \ldots, \beta_n)}{\partial(\beta, \alpha_1, \ldots, \alpha_{n-1})} \right| \mathrm{d}\beta \prod_{i=1}^{n-1} \mathrm{d}\alpha_i.$$

$$\Rightarrow \prod_{i=1}^{n} \mathrm{d}\beta_i = \beta^{\,n-1}\, \mathrm{d}\beta \prod_{i=1}^{n-1} \mathrm{d}\alpha_i.$$

$$\Rightarrow \prod_{i=1}^{n} \mathrm{d}\beta_i = \beta^{\,n-1}\, \mathrm{d}\beta \prod_{i=1}^{n} \mathrm{d}\alpha_i\; \delta\Big(1 - \sum_{i=1}^{n} \alpha_i\Big).$$

Therefore, the Schwinger-parameter representation is-

$$f(p) = \# \int_0^\infty \mathrm{d}\beta\; \beta^{\,n-1} \int_0^1 \left(\prod_{i=1}^{n} \mathrm{d}\alpha_i \right) \delta\Big(1 - \sum_{i=1}^{n} \alpha_i\Big)\; \frac{e^{-\tilde{\nu}(\beta, \alpha_i)}}{\phi(\beta, \alpha_i)^{-mD/2}}$$

Here ν and ϕ are homogeneous function of degree one, then,

$$\psi(\beta,\alpha_i) = \beta\,\psi'(\alpha_i), \qquad \phi(\beta,\alpha_i) = \beta\,\phi'(\alpha_i),$$

where ν', ϕ' are functions of the α_i only.

f(p) becomes,

$$\Rightarrow f(p) = \# \int_0^\infty \mathrm{d}\beta\; \beta^{n-1} \int_0^1 \left(\prod_{i=1}^n \mathrm{d}\alpha_i\right) \delta\Big(1 - \sum_{i=1}^n \alpha_i\Big)\; \frac{e^{-\beta\,\nu'(\alpha_i)}}{\big(\beta\,\phi'(\alpha_i)\big)^{-mD/2}}$$

$$\Rightarrow f(p) = \# \int_0^\infty \mathrm{d}\beta\; \beta^{n-1}\; \beta^{-mD/2} \int_0^1 \left(\prod_{i=1}^n \mathrm{d}\alpha_i\right) \delta\Big(1 - \sum_{i=1}^n \alpha_i\Big)\; \frac{e^{-\beta\,\nu'(\alpha_i)}}{(\phi'[\alpha_i)]^{-mD/2}}$$

$$\Rightarrow f(p) = \# \textstyle\int_0^\infty \mathrm{d}\beta\; \beta^{n-\frac{mD}{2}-1}\; e^{-\beta\,\nu'(\alpha_i)} \int_0^1 \left(\prod_{i=1}^n \mathrm{d}\alpha_i\right) \delta\Big(1 - \sum_{i=1}^n \alpha_i\Big)\; \frac{1}{[\phi'(\alpha_i)]^{-mD/2}}$$

$$\Rightarrow f(p) = \#\, \Gamma\Big(n - \tfrac{mD}{2}\Big) \int_0^1 \left(\prod_{i=1}^n \mathrm{d}\alpha_i\right) \delta\Big(1 - \sum_{i=1}^n \alpha_i\Big)\; \frac{1}{\big(\phi'(\alpha_i)\big)^{mD/2} \big(\nu'(\alpha_i,p_i)\big)^{\left(n-\frac{mD}{2}\right)}}$$

We can write this as,

$$f(p_{jk}) = \lim_{\epsilon\to 0^+} \int_0^1 d\alpha_1 ... d\alpha_n \frac{\phi(\alpha_i)\delta(\alpha_1 + ... + \alpha_n - 1)}{[F'(\alpha_i;p_{jk}) + i\epsilon]^{n-2m}}$$

where we considered D = 4 and we re-labeled the variables according to the paper.

The function F' is obtained from F by taking its stationary values with respect to the integration variables. The function $\phi(\alpha_i)$ is a simple algebraic function of the α's, relevant only when it vanishes at some critical value of the α_i.

The δ-function can be eliminated and so the integral becomes :

$$f(z_{jk}) = \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 ... \int_0^{1-\alpha_1-...-\alpha_{n-1}} d\alpha_{n-1} \frac{1}{[F''(\alpha_i,z_{jk})]^{n-2m}}$$

. We define $F'' = F' - \lambda(\sum_i \alpha_i - 1)$

Let F denote the original Feynman denominator and F' the form obtained after integrating over the loop momenta. We impose the normalization condition

$$\sum_i \alpha_i = 1.$$

To determine F', we define

$$F'' = F' - \lambda\left(\sum_i \alpha_i - 1\right),$$

where λ is a Lagrange multiplier enforcing the constraint. The stationary condition with respect to α_i gives

$$\frac{\partial F''}{\partial \alpha_i} = \frac{\partial F'}{\partial \alpha_i} - \lambda = 0,$$

which implies

$$\frac{\partial F'}{\partial \alpha_i} = \lambda.$$

Multiplying both sides by α_i and summing over i, we obtain

$$F' = \sum_i \alpha_i \frac{\partial F'}{\partial \alpha_i} = \lambda \sum_i \alpha_i = \lambda.$$

Hence,

$$\boxed{F' = \lambda.}$$

Now, consider the explicit form of F:

$$F = \sum_i \alpha_i \left(q_i^2 - m_i^2\right),$$

where q_i are the momenta associated with each internal line. Since F' is obtained from F by taking its stationary value with respect to the loop momenta k_j, we can write

$$F'(\alpha, p) = F(\alpha, k', p),$$

where k'_j denotes the stationary configuration of F with respect to k_j. Thus,

$$\frac{\partial F'}{\partial \alpha_i} = \frac{\partial F}{\partial \alpha_i} + \frac{\partial F}{\partial k_j}\frac{\partial k'_j}{\partial \alpha_i}.$$

At the stationary point, $\frac{\partial F}{\partial k_j} = 0$, which leads to

$$\frac{\partial F'}{\partial \alpha_i} = \frac{\partial F}{\partial \alpha_i} \qquad \text{(evaluated at } k = k'\text{)}.$$

Since

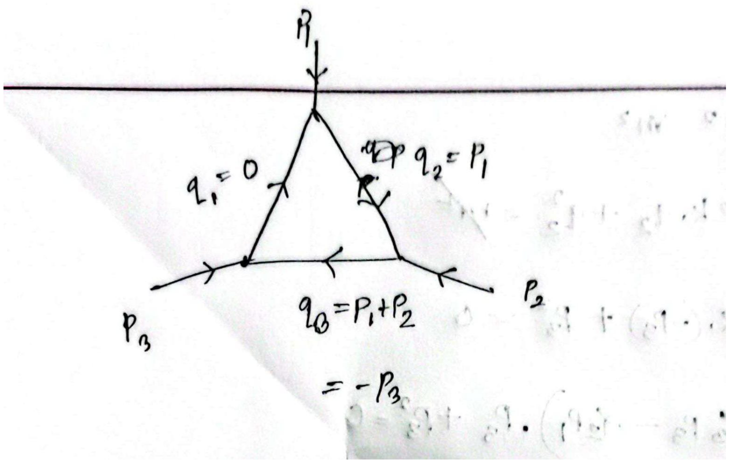
$$F = \sum_j \alpha_j (q_j^2 - m_j^2),$$

we finally have

$$\frac{\partial F'}{\partial \alpha_i} = q_i^2 - m_i^2.$$

The Landau conditions then read

$$\boxed{\begin{array}{l} \alpha_i \left(q_i^2 - m_i^2\right) = 0, \quad i = 1, \ldots, n, \\ F'(\alpha; p) = 0. \end{array}}$$



3 Solving Landau Equation for One-Loop Triangular Diagram

$$-\sum_i \alpha_i q_i = k$$
$$\Rightarrow -(\alpha_2 p_1 + \alpha_3(-p_3)) = k$$
$$\Rightarrow k = \alpha_3 p_3 - \alpha_2 p_1 \tag{8}$$

and

$$(k + q_1)^2 = m^2$$
$$\Rightarrow k^2 = m^2 \tag{9}$$

Again,

$$(k + p_1)^2 = m^2 \tag{10}$$
$$\Rightarrow 2\alpha_2 p_1^2 - 2\alpha_3 p_3 \cdot p_1 = p_1^2$$

Similarly,

$$(k + q_2)^2 = m^2$$
$$-2\alpha_2 p_1 \cdot p_3 + 2\alpha_3 p_3^2 + p_3^2 = 0 \tag{11}$$

Solving these two equations we get ,

$$\alpha_2 = \frac{p_3^2(p_1^2 + p_1 \cdot p_3)}{2p_1^2 p_3^2 - 2(p_1 \cdot p_3)^2}$$

and

$$\alpha_3 = \frac{p_1^2(p_3^2 + p_1 \cdot p_3)}{2p_3^2 p_1^2 - 2(p_1 \cdot p_3)^2}$$

Now taking square of (8) and from the equation (9)

$$k \cdot k = \alpha_3^2 p_3^2 + \alpha_2^2 p_1^2 - 2\alpha_2\alpha_3(p_1 \cdot p_3) \tag{12}$$
$$\Rightarrow k^2 = \frac{(p_1^2)^2 p_3^2 (p_3^2 + p_1 \cdot p_3)^2 + (p_3^2)^2 p_1^2 (p_1^2 + p_1 \cdot p_3)^2 - 2p_1^2 p_3^2 (p_1 \cdot p_3 (p_1^2 + p_1 \cdot p_3)(p_3^2 + p_1 \cdot p_3)}{4(p_1^2 p_3^2 - (p_1 \cdot p_3)^2)^2}$$

$$\Rightarrow m^2 = \frac{p_1^2 p_3^2 (p_1^2 p_3^4 + p_3^2 p_1^4 + 2p_1^2 p_3^2 (p_1 \cdot p_3) - p_1^2 (p_1 \cdot p_3)^2 - p_3^2 (p_1 \cdot p_3)^2 - 2(p_1 \cdot p_3)^3}{4(p_1^2 p_3^2 - (p_1 \cdot p_3)^2)^2}$$

$$\Rightarrow m^2 = \frac{p_1^2 p_3^2 (p_1^2 p_3^2 (p_1^2 + p_3^2 + 2(p_1 \cdot p_3)) - (p_1 \cdot p_3)^2 (p_1^2 + p_3^2 + 2(p_1 \cdot p_3))}{4(p_1^2 p_3^2 - (p_1 \cdot p_3)^2)^2}$$

$$\Rightarrow m^2 = \frac{p_1^2 p_3^2 (p_1^2 + p_3^2 - 2(p_1 \cdot p_3))}{4(p_1^2 p_3^2 - (p_1 \cdot p_3)^2)}$$

$$\Rightarrow \boxed{p_1^2 p_3^2 (p_1^2 + p_3^2 - 2(p_1 \cdot p_3)) = 4m^2} \tag{13}$$

which is actually the Landau equation denoting a surface, defining the limit of analiticity.

4 Outlook

In this work, we examined how singularities in Feynman amplitudes arise using the Landau equations. These singularities appear when internal lines go on-shell together, forming pinch surfaces that define the limits of analyticity.

There are several directions for further exploration. Extending this analysis to more complex diagrams, higher-loop cases, or non-planar amplitudes could uncover richer structures and overlapping singularities. A more detailed study of the regions where amplitudes remain analytic, possibly relaxing some of the assumptions used here, would also deepen our understanding of scattering amplitudes in quantum field theory.

Beyond the analytic approach, recent developments suggest promising algebraic methods for understanding Feynman integrals. For example, constructing differential operators that annihilate specific families of integrals, like the ℓ-loop banana diagrams, provides a new perspective on their algebraic structure. Investigating whether similar operator structures exist for other diagram families could reveal underlying symmetries or geometric principles, potentially offering an alternative to traditional integration-by-parts techniques.

Overall, combining the insights from Landau analysis with these algebraic approaches offers a powerful framework for connecting the mathematics of Feynman integrals with their physical interpretation. Pursuing these directions further may uncover some new features.

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Approval

The internship report titled "**Properties of Scattering Amplitudes**" submitted by **Nusrat Jahan Priti**, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfilment of the requirements of the internship program.

The internship was conducted under the supervision of **Ratul Mahanta** during the period **15 July 2025** to **15 October 2025**.

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