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A Guide to Analytic Structure of Feynman Amplitudes and Landau Singularities

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Chapter 1

Abstract

In perturbative quantum field theory, Feynman diagrams encode all possible particle interactions, but evaluating their loop integrals often leads to analytic complications—branch points and poles in the amplitudes as functions of external momenta or masses. Landau's analysis shows that these singularities arise precisely when certain internal propagators go on shell (their momenta satisfy ($q_i^2 = m_i^2$)) and when the integration contour in the complex Feynman–parameter space becomes pinched by coincident zeros of the combined denominator. In this report we review Polkinghorne and Screaton's method for locating all singularities of multi-loop Feynman diagram amplitudes via contour deformations in Feynman–parameter space. We derive the general Landau equations—($\partial M/\partial \alpha_i = 0$) together with ($M = 0$) and the requirement that each (α_i) either vanish or multiply an on-shell condition—and illustrate their use with one-loop triangle and box examples. We then summarize the explicit analytic structure of the third-order vertex function, showing how branch cuts and anomalous thresholds emerge when square-root or logarithmic arguments vanish.

Chapter 2

Introduction

In quantum field theory (QFT), particles do not resemble little billiard balls colliding with one another. They are excitations of fundamental fields, which continuously interact, transferring energy and momentum in many tiny manners. We use perturbation theory to expand the interaction in powers of a small coupling constant. Through the Dyson series and Wick's theorem, we convert time-ordered products of fields into sums of contractions, each corresponding to propagators connecting field points. When these algebraic contractions are shown, they transform into **Feynman diagrams**—schematic yet mathematically exact representations of all potential interactions among particles. By taking into account the higher order of the perturbative expansion of the Green Functions, then we encounter with the feature of the loop diagram. Each loop introduces an integration over undetermined internal momenta. In practice, evaluating these integrals is not straightforward. These loop integrals often lead to divergences at the poles of the propagator. When those loop integrals are written out, the denominators from propagators turn the Feynman amplitude into a complex function of external momenta. This is where the **analyticity of Feynman diagram** discussion comes.

Our main discussion of this work is related to this topic. Analyticity tells us the amplitude is smooth almost everywhere; singularities mark the places where real physical events emerge from internal line ones; and Landau's analysis maps those singularities geometrically. We will focus in this report about the **Landau equations** that identify the conditions under which the denominatos and momentum-conservation constraints simultaneously produce a pinch singularity.

Mainly, in this report we are reviewing the papaer work [1]. Polkinghorne and Screaton present a contour-deformation method to locate all singularities of Feynman-diagram amplitudes, showing that they arise when Feynman parameters hit endpoints or when zeros of the denominator pinch the integration contour. These conditions correspond to the Landau equations, requiring internal lines to be on- or off-shell and weighted sums of momenta in loops to vanish. They illustrate this with one-loop examples and analyze thresholds of a third-order vertex diagram.

Chapter 3

Analyticity of Feynman diagram

3.1 Feynman-parameter representation and analyticity

Feynman loop diagrams in quantum field theory can be viewed as contour integrals in the complex plane, where the integration runs over the internal (loop) momenta. The resulting amplitudes are analytic functions of the external variables, such as energy or momentum invariants, except at points where physical singularities occur. Consider a generic 'l-loop Feynman integral with n propagators in D-dimensions, which takes the form:

$$I(p_i) = \int \frac{\mathrm{d}^D k_1 \mathrm{d}^D k_2 \cdots \mathrm{d}^D k_\ell}{A_1 A_2 \cdots A_n}. \tag{3.1}$$

where,

$$A_i = m_i^2 - q_i^2$$

The propagators A_i are quadratic in the external momenta p_i and loop momenta k_i. When combined with Feynman parameters α_i, this integral can be rewritten as:

$$I(p_i) = (n-1)! \int \frac{d^D k_l d^n \alpha}{M^n} \delta\Big(1 - \sum_{i=1}^{n} \alpha_i\Big) \tag{3.2}$$

where,

$$M = \alpha_1 A_1 + \cdots + \alpha_n A_n.$$

The integral $I(p_i)$ can develop a singularity only when the first polynomial $M(\{\alpha_j\}, p_i)$ vanishes somewhere on the Feynman-parameter integration region, and each Feynman variable α_i either sits at the endpoint $\alpha_i = 0$ or lies on a factor $A_i = 0$ of the denominator. If all $\alpha_i \neq 0$, the solution of $M = 0$ together with the simultaneous vanishing of all partial derivatives $\partial M/\partial\alpha_k = 0$ defines the leading singularity; if instead a subset of the α_i vanishes, one obtains a subleading singularity. However, merely satisfying α_i (and some α_i) at a generic point does not force a divergence, since the integration contour can be deformed around that zero. A genuine singularity arises only when the contour is pinched by two or more coincident zeros of the denominator—equivalently when both $M = 0$ and $\partial M/\partial\alpha_k = 0$ hold for each active α_k. Together with the conditions that each external invariant either sets its conjugate $\partial M/\partial p_i$ to zero or drives one of the α_i to zero, these pinching requirements form the complete set of Landau equations [2]. Now, in the next section we will see the detailed discussion of the singularities and Landau equations.

3.2 Singularities

We can represent the integral 3.2 by considering an arbitrary Feynman diagram $G(p_s^\mu)$ with external momentum (p_s^μ):

$$G(\{p_s^\mu\}) = \prod_{\text{lines } i} \int_0^1 d\alpha_i \delta\big(\sum_i \alpha_i - 1\big) \times \prod_{\text{loops } r} \int d^n k_r D(\alpha_i, k_r, p_s)^{-N} F(\alpha_i, k_r, p_s). \tag{3.3}$$

Where,

$$D(\alpha_i, k_r, p_s) = \sum_j \alpha_j \left[\ell_j^2(p,k) - m_j^2\right] + i\varepsilon. \tag{3.4}$$

$F(\alpha_i, k_r, p_s)$ is the constant and numerator factor. α_j is the Feynman parametre of the jth line. And $\ell_j(p,k)$ is the momentum which is a linear function of the loop momenta (k_r) and externam momenta (p_s). In the absense of singularities, $G(p_s^\mu)$ is analytical function of (p_s^μ). Singularities arise from the zeroes of the denomination $D(\alpha_i, k_r, p_s)$.

Consider an isolated pole $P(Z)$ in a complex plan in any one of the integration variables $(\zeta \in \{k_r^\mu, \alpha_i\})$. The isolated poles $p(Z)$ can be avoided by deforming the ζ contour. In this new contour, the integral can be analytical in the external momentum. But, deformation of the contour is not always possible. In this case, there are two other ways the integration can form singularitis [3]. Those are:

- End-point singularity: Consider a general function $F(\zeta, w)$ of a variable w. The complex variable ζ of this function has fixed end points ζ_a and ζ_b. Integral over the complex variables of the function can be written as:

$$I(w) = \int_{\zeta_a}^{\zeta_b} d\zeta F(\zeta, w). \tag{3.5}$$

$F(\zeta, w)$ standa for D^{-N}. Isolated poles position $\xi(w)$ depends on the variable w. Lets say $\xi(w_0) = \zeta_a$ for $w = w_0$. Then the pole is at one of the end-points of the integral and $I(w)$ is undefined. This is called an end-point singularity.

- Pinch singularities: When two or more pole $\xi_i(w)$ and $\xi_{i'}(w)$ approches to the contour ζ from opposite side and conside at point w_0 so $\xi_i(w_0) = \xi_{i'}(w_0)$. The countour trapped between them and no deformation can avoid them. So, we can say, this gives a pinch singularity at $w = w_0$.

3.3 Landau equations

Landau equations emerges by analyzing when and where the denominator and its derivative vanish together [4]. Which is mainly represents the condition for the pinch singularity. Mainly, Landau equations give us the location of all the singularities of the Feynman diagrams. There are two main forms of these equations, depending on which representation of the Feynman integral is studied. We already know when 3.4, the denominator is zero, the singularity emerges and The denominator is quadratic in loop momentum k_r^μ. This we can represent as:

$$\sum_{\text{lines } i} \alpha_i \left[l_i^2(p,k) - m_i^2 \right] + i\epsilon = 0 \tag{3.6}$$

$$l_i^\mu(p,k) = \sum_r \eta_{ir} k_r^\mu + \sum_s \hat{\eta}_{is} p_s^\mu \tag{3.7}$$

Here, η_{ir} and $\hat{\eta_{is}}$ are incidence metrices for loop and external momentum. For each loops j, that includes the lines i:

$$0 = \frac{\partial}{\partial k_j^\nu} \Big\{ \sum_{\text{lines } i} \alpha_i \left[l_i^2(p,k) - m_i^2 \right] + i\epsilon \Big\} = 2 \sum_{\text{lines } i} \eta_{ij} \big(\alpha_i l_i^\mu \big). \tag{3.8}$$

The denominator $D(\alpha_i, k_r, p_s)$ will be independent of α_i if:

$$\ell_i^2(p,k) = m_i^2 \quad \text{or} \quad \alpha_i = 0 \tag{3.9}$$

Together 3.8 and 3.9 represnts the Landau equations. From Landau equations, it is possible to assign a physical interpretation to the case of Feynman diagrams and amplitudes. [5] showed that whenever the Landau equations are satisfied, one can literally picture the Feynman diagram as a set of classical particles ("on-shell lines") flying between interaction points in space–time. Each Feynman parameter (α_i) plays the role of the Lorentz-invariant ratio of "propagation time" to energy for particle (i), so that the space–time separation between two vertices (a) and (b) is

$$x_b^\mu - x_a^\mu = \sum_{i \in \text{path}} \alpha_i \ell_i^\mu.$$

The requirement is that every path between the same pair of vertices gives the same (Δx^μ) - is precisely equivalent to the momentum-flow (loop) conditions $(\sum_i \alpha_i\, \ell_i^\mu = 0)$. In other words, only when the Landau equations hold can one interpret the diagram as a set of freely propagating on-shell particles meeting at definite space–time points.

In the other case of Landau equations, whenever a propagator line is off its mass shell $(\ell_i^2 \neq m_i^2)$, the corresponding Feynman parameter (α_i) must be zero. Physically, that means the particle doesn't really "travel" between vertices, so that we can collapse its line to a point. Doing this for every off-shell line turns the full diagram into a simpler "reduced" graph where only the on-shell (physical) lines remain stretched out between interaction points. These **reduced diagram** make it much easier to spot exactly which configurations of on-shell particles pinch the integration contour and produce singularities.

Regarding the reduced diagram, another point that naturally arises in the discussion is the leading and lower-order singularities [6]. When a diagram hits a singularity, every internal line is either "on shell" (its momentum satisfies the mass-shell condition) or its Feynman parameter (α_i) is zero. If $(\alpha_i = 0)$, that line carries no physical propagation and can be collapsed to a point—this contraction doesn't change momentum conservation, so the singular behavior is the same as in the smaller diagram where that line is removed. The singularity involving only on-shell lines (no $(\alpha_i = 0)$ is called the **leading singularity**, because it comes from the full set of physical propagators. Any singularity obtained after collapsing one or more lines $(\alpha_i = 0)$ is a **lower-order singularity**, shared by the reduced graphs.

3.3.1 Triangle graph

We can consider a one-loop three-point (triangle) graph case to illustrate the application of Landau singularities. Taking $p_1, p_2 \text{and} p_3$ as the external momentum, which satisfy:

$$p_1 + p_2 + p_3 = 0 \tag{3.10}$$

For the one-loop three-point (triangle) graph with all internal and external masses equal to m, and with Feynman parameters $\alpha_1, \alpha_2, \alpha_3$ satisfying:

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 \tag{3.11}$$

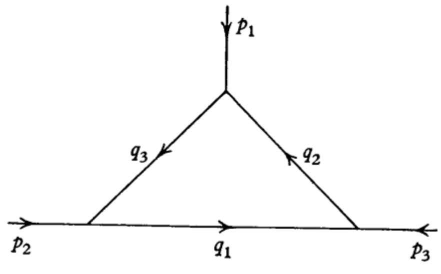

Figure 3.1: Momentum-flow in triangle case.

Let us choose the loop momentum k and the three internal momenta q_1, q_2, q_3 can be represent as:

$$q_1 = k, \tag{3.12}$$
$$q_2 = k + p_1, \tag{3.13}$$
$$q_3 = k - p_3 \tag{3.14}$$

the Landau equations (with no $\alpha_i \neq 0$ solutions) are simply:

- On-shell condition for each internal line i:

$$\sum_i \alpha_i (q_i^2 - m^2) = o \tag{3.15}$$

  as $\alpha_i \neq 0$ so,

$$q_i^2 = m^2, \; i = 1, .., 3 \tag{3.16}$$

  putting 3.12-3.14 values into 3.16, we found:

$$k^2 = m^2 \tag{3.17}$$

$$\begin{aligned} (k + p_1)^2 &= m^2 \\ k^2 + 2k.p_1 + p_1^2 &= m^2 \\ 2p_1.k + p_1^2 &= 0 \end{aligned} \tag{3.18}$$

$$\begin{aligned} (k - p_3)^2 &= m^2 \\ k^2 - 2k.p_3 + p_3^2 &= m^2 \\ p_3^2 - 2p_3.k &= 0 \end{aligned} \tag{3.19}$$

- Momentum-derivative condition:

$$\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 = 0 \tag{3.20}$$

now again put the internal momenta in 3.20:

$$\begin{aligned}
\alpha_1 k + \alpha_2(k + p_1) + \alpha_3(k - p_3) &= 0 \\
(\alpha_1 + \alpha_2 + \alpha_3)k + \alpha_2 p_1 - \alpha_3 p_3 &= 0 \\
k &= \alpha_3 p_3 - \alpha_2 p_1
\end{aligned} \tag{3.21}$$

Substituiting this k in the 3.18 and 3.19:

$$\begin{aligned}
p_1^2 + 2p_1 \cdot (\alpha_3 p_3 - \alpha_2 p_1) &= 0 \\
2p_1^2 \alpha_2 - 2(p_1 \cdot p_3)\alpha_3 &= p_1^2
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
p_3^2 - 2p_3 \cdot (\alpha_3\, p_3 - \alpha_2 p_1) &= 0 \\
2(p_1 \cdot p_3)\alpha_2 - 2p_3^2 \alpha_3 &= -p_3^2
\end{aligned} \tag{3.23}$$

As here in 3.22 and 3.23 we can see the linear expression of the α's, then by considering 2×2- matrix, we can find the equation for α_2 and α_3. Which can be expressed as:

$$\begin{aligned}
\alpha_2 &= \frac{p_1^2\,(-2p_3^2) - (-p_3^2)(-2\,(p_1{\cdot}p_3))}{-2\,p_1^2\,p_3^2 + 2\,(p_1{\cdot}p_3)\,2\,(p_1{\cdot}p_3)} \\
&= \frac{p_1^2\,p_3^2 + p_3^2\,(p_1{\cdot}p_3)}{2\,p_1^2\,p_3^2 - 2\,(p_1{\cdot}p_3)^2} \\
&= \frac{p_3^2\,\left(p_1^2 + p_1{\cdot}p_3\right)}{2\,p_1^2\,p_3^2 - 2\,(p_1{\cdot}p_3)^2} = \frac{p_3^2\,p_1{\cdot}(p_1 + p_3)}{2\,p_1^2\,p_3^2 - 2\,(p_1{\cdot}p_3)^2} \\
&= \frac{p_3^2\,(p_1{\cdot}p_2)}{2\,(p_1{\cdot}p_3)^2 - 2\,p_1^2\,p_3^2}
\end{aligned} \tag{3.24}$$

and,

$$\begin{aligned}
\alpha_3 &= \frac{2\,p_1^2\,(-p_3^2) - 2\,(p_1{\cdot}p_3)\,p_1^2}{2\,p_1^2\,(-2\,p_3^2) + 4\,(p_1{\cdot}p_3)^2} \\
&= \frac{p_1^2\,p_3^2 + (p_1{\cdot}p_3)\,p_1^2}{2\,p_1^2\,p_3^2 - 2\,(p_1{\cdot}p_3)^2} = \frac{p_1^2\,(p_2{\cdot}p_3)}{2\,(p_1{\cdot}p_3)^2 - 2\,p_1^2\,p_3^2}
\end{aligned} \tag{3.25}$$

Now doing the square of the 3.21:

$$\begin{aligned} k^2 &= (\alpha_3\, p_3 - \alpha_2\, p_1) \cdot (\alpha_3\, p_3 - \alpha_2\, p_1) \\ &= \alpha_3^2\, p_3^2 - \alpha_2\, \alpha_3\, (p_1 {\cdot} p_3) - \alpha_3\, \alpha_2\, (p_3 {\cdot} p_1) + \alpha_2^2\, p_1^2 \\ &= \alpha_3^2\, p_3^2 - 2\, \alpha_2\, \alpha_3\, (p_1 {\cdot} p_3) + \alpha_2^2\, p_1^2. \end{aligned} \tag{3.26}$$

Now taking the equation 3.17 and putting the value of α_2 and α_3:

$$\begin{aligned} k^2 &= m^2 \\ \alpha_3^2\, p_3^2 - 2\, \alpha_2\, \alpha_3\, (p_1 {\cdot} p_3) + \alpha_2^2\, p_1^2 &= m^2 \end{aligned} \tag{3.27}$$

$$\left(p_1^2\right)^2 (p_1 {\cdot} p_3)^2 (p_3)^2 - 2\, p_3^2 (p_1 {\cdot} p_2)\, p_1^2 (p_3 {\cdot} p_3)(p_1 {\cdot} p_3) + (p_3^2)^2 (p_1 {\cdot} p_2)\, p_1^2 = 4\, m^2 \left[p_3^2 p_1^2 - (p_1 {\cdot} p_3)^2\right]^2 \tag{3.28}$$

This is the Landau surface for the one-loop triangle diagram:(p_2, p_3)-surface, where the integral can develop a singularity. We can see the Landau Singularities application in the Square-box case too by following the same calculations, which is broadly done in the Appendix A.

Chapter 4

The Analytic Properties of Perturbation Theory

As mentioned earlier, our main focus here is to examine the work of Polkinghorne and Screaton [1] on the analytic properties of perturbation theory. The necessary background material and theoretical groundwork required for this review have already been covered in the previous section, so we can now move directly into the discussion of their contributions and findings.

4.1 Perturbation-Theoretic Functions and Supporting Lemmas

In perturbative quantum field theory, each Feynman graph with n-internal lines and m-loops is first rewritten by introducing n-nonnegative Feynman parameters $\alpha_1, ..., \alpha_n$ and a delta function $\delta(\sum \alpha_n - 1)$ to combine all propagator denominators into a single factor:

$$\lim_{\varepsilon \to 0^+} \int_0^\infty d\alpha_1 \cdots d\alpha_n \delta\big(\alpha_1 + ... + \alpha_n - 1\big) \int d^4k_1 \cdots d^4k_m \frac{1}{[F(\alpha_i, k_j, p_k) + i\varepsilon]^n} \tag{4.1}$$

Where,

$$F(\alpha_i; k_j; p_k) = \sum_{i=1}^{n} \alpha_i \big(q_i^2 - m_i^2\big) \tag{4.2}$$

collects all quadratic forms in the loop momenta (k_j), the external momenta (p_k), and the masses (m_i). After regularizing and performing the (k_j)-integrals by symmetric (Euclidean) integration, one obtains a purely Feynman-parameter integral(The detailed calculation for this part has already been presented in Appendix

B, so here we will focus only on the main results and their implications.)

$$f(p_{jk}) = \lim_{\varepsilon \to 0^+} \int_0^\infty d\alpha_1 \cdots d\alpha_n \delta\big(\sum \alpha_i - 1\big) \frac{\varphi(\alpha_i)}{\big[F'(\alpha_i; p_{jk}) + i\varepsilon\big]^{n-2m}}, \tag{4.3}$$

where $(F'(\alpha_i; p_{jk}))$ is the "reduced" denominator found by solving the stationarity conditions

$$\frac{\partial F}{\partial k_j} = 0 \quad (j = 1, \ldots, m)$$

and substituting back. This procedure both consolidates denominators into a single algebraic function of the (α_i) and exposes the analytic structure and singularities of the amplitude via the critical points of (F').

When one drops the infinitesimal $(i\varepsilon)$ shift on the combined denominator, the resulting parametric integral naturally extends to complexified internal masses or invariants (z_{jk}). Thus the amplitude becomes

$$f(z_{jk}) = \int_0^\infty d\alpha_1 \cdots d\alpha_n \delta\big(\textstyle\sum_i \alpha_i - 1\big) \frac{\varphi(\alpha)}{\big[F'(\alpha; z_{jk})\big]^{n-2m}}, \tag{4.4}$$

where $(F'(\alpha; z_{jk}))$ is the reduced quadratic form obtained by stationary-phase conditions $(\partial F/\partial k_j = 0)$. Analytic continuation in the complex parameters (z_{jk}) then allows one to trace the location and nature of singularities (branch points or poles) by studying where $(F' = 0)$ on the integration domain of the Feynman parameters. To analyze the singularities of this function, it is useful first to establish a lemma inspired by the work of Eden (10).

4.1.1 Lemma

The lemma states that a function [7]

$$f(\zeta) = \int_C F(u, \zeta) du \tag{4.5}$$

which is analytic in a domain (Z) (because for each fixed $(\zeta \in Z)$ the integrand $(F(u, \zeta))$ has no singularities on the contour (C), can be continued analytically along any path in the (ζ)-plane that does not force one of the moving singularities $(u_i(\zeta))$ of (F) to cross the contour. Concretely:

• As (ζ) moves from an initial point $(\zeta_0 \in Z)$ along a path to some (ζ_1) outside (Z), each singularity $(u_i(\zeta))$ traces out a continuous path in the (u)-plane (shown in the figure).

• We then deform the original contour (C) into a new contour (C_1) that avoids all these moving singularities.

• The integral

$$f_1(\zeta) = \int_{C_1} F(u, \zeta) du \tag{4.6}$$

defines an analytic continuation of $(f(\zeta))$ to (ζ_1). By repeating this process step by step, one can analytically continue $(f(\zeta))$ along any desired route, so long as no singularity ever crosses the integration contour. As discussed earlier in Section 3.2, singularities can arise through mechanisms such as endpoint singularities and pinch singularities. We use this lemma because our Feynman-parameter integral 4.4 stays analytic in the complex variable (z) only if the zeros or branch points of $(F'(\alpha; z) = 0)$ do not enter the integration region in (α)-space. The lemma shows how we can smoothly deform this region to follow the motion of singularities without crossing them, allowing $(f(z))$ to be analytically continued in the external invariants or masses.

4.2 Singularities in multiple integral case

If we now consider the case of a multiple integral, the expression takes the following form: From 4.4remove the delta function $(\delta(\alpha_1 + \cdots + \alpha_n - 1))$ by solving for $(\alpha_n = 1 - (\alpha_1 + \cdots + \alpha_{n-1}))$. This reduces the full Feynman-parameter integral to an $((n-1))$-fold integral, where each(α_i) runs between 0 and 1-the sum of the earlier (α)'s:

$$f(z_{jk}) = \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \cdots \int_0^{1-\alpha_1-\cdots-\alpha_{n-2}} d\alpha_{n-1} \frac{1}{[F''(\alpha_i; z_{jk})]^{n-2m}} \tag{4.7}$$

Here, (F'') is simply (F') with (α_n) removed. This ordering of integrations temporarily breaks the symmetry among the (α)'s, though the final expression can be made symmetric again. Singularities in (f) appear only when an (α)-variable reaches its boundary (0 or 1) or when it coincides with another zero of the denominator. Showing this rigorously, however, requires applying the analytic-continuation lemma step by step, since each integration can make the remaining function multi-valued.

In the multiple integration case 4.7, each integration can give rise to either an endpoint singularity or a coincident singularity. Therefore,

$$F''(\alpha_i, z_{ij}) = 0 \tag{4.8}$$

together with: Either:

i) $\alpha_i = 0$ (lower limit end point) or $\alpha_n = 0$ (upper limit end point);

ii) $\dfrac{\partial F''}{\partial \alpha_i} = 0$ and the resulting coincident singularities in the α_i integration "pinch" the contour;

for $i = 1, \ldots, n-1$.

By imposing the overall constraint $\alpha_1 + ... + \alpha_n = 1$, $F''(\alpha_i, z_{jk})$ can be replaced by $F'(\alpha_i, z_{jk})$ which is homogeneous of degree one in the α's. Enforces that constraint via a Lagrange multiplier λ:

$$\frac{\partial F''}{\partial \alpha_i} = \frac{\partial F'}{\partial \alpha_i} + \lambda = 0, \quad i = 1, \ldots, n, \tag{4.9}$$

Euler's theorem for homogeneous functions of degree one gives:

$$\sum_i^n \alpha_i \frac{\partial F'}{\partial \alpha_i} = F'(\alpha_i, z_{jk}). \tag{4.10}$$

Vanishing of $F'(\alpha_i, z_{jk})$ implies that $\lambda = 0$ with the other conditions:

$$\begin{cases} \text{either:} \quad \alpha_i = 0, \\ \text{or:} \quad \dfrac{\partial F'}{\partial \alpha_i} = 0, \end{cases} \tag{4.11}$$

These conditions are just the generalization of the condition given by [4] for the real singularities and derived by him in a different manner. Which follows that:

$$\frac{\partial F'}{\partial \alpha_i} = \left(q_i^2 - m_i^2\right) + \sum_{j=1}^{m} \frac{\partial F}{\partial k_j} \frac{\partial k_j}{\partial \alpha_i} = \left(q_i^2 - m_i^2\right), \tag{4.12}$$

this shows these Landau equations amount to "either the ith propagator is off ($\alpha_i = 0$), or it is on shell ($q_i^2 = m_i^2$) and all loop-momentum derivatives vanish." The restriction on coincident singularity, which prevents pinching as described in equation 4.11, excludes certain harmless coincidences. This means that not all points satisfying the weaker conditions are actually relevant.

$$\begin{cases} \text{either:} \quad \alpha_i = 0, \\ \text{or:} \quad \dfrac{\partial F}{\partial \alpha_i} = 0, \end{cases} \tag{4.13}$$

These points are not singularities of (f)on the physical sheet defined by the integral representation 4.7. However, when the function is analytically continued to all its Riemann sheets [8], equations 4.13 do identify the possible singular points. This happens because the continuation can be viewed as considering all possible integration contours connecting the same endpoints—so any coinciding singularities will inevitably pinch one of those contours.

4.2.1 One-loop diagram

For any one-loop Feynman graph with n internal lines, the reduced denominator after combining all propagators and enforcing $\sum_{i=1}^{n} \alpha_i = 1$ can be written as a quadratic form in the Feynman parameters plus a trivial linear term. For the contraint $\sum_{i=1}^{n} \alpha_i = 1$, only important part is considered important is the matrix part:

$$\sum_{i,j} A_{ij}(z)\, \alpha_i\, \alpha_j$$

A nontrivial solution $\{\alpha_i\} \neq 0$ to these linear equations exists exactly when the matrix $A_{ij}(z)$ is singular. Equivalently,

$$\boxed{\det\big[A_{ij}(z)\big] = 0}. \tag{4.14}$$

This single condition is the criterion for the pinch singularity in the one-loop diagram. The other singularities, corresponding to the vanishing of one or more α's, are then given by the vanishing of the principal minors of $\det\big[A_{ij}(z)\big]$ obtained by omitting the appropriate rows and columns [9].

4.3 The third-order vertex function

We now discuss to an example that illustrates how singularities can appear and disappear on different sheets—the third-order vertex function, which is the main focus of this paper. By considering the equality of all masses in the internal particles, we take the differential third-order vertex function [10]:

$$f(Z, Z_3) = \int_0^1 d\alpha d\beta d\gamma \frac{\delta(1-\alpha-\beta-\gamma)}{\big[m^2 - (\alpha+\beta)\gamma Z - \alpha\beta Z_3\big]^2} \tag{4.15}$$

This function is an example of a one-loop diagram and therefore, the condition 4.14 is applicable here to find the possible singularities. Now we can work a little on the integration function 4.15:

First, use the δ-function to do the γ-integral immediately. The constraint $\delta(1-\alpha-\beta-\gamma)$ sets

$$\gamma = 1 - \alpha - \beta,$$

and removes that integration, giving

$$f(Z, Z_3) = \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{\big[m^2 - (\alpha+\beta)Z - \alpha\beta Z_3\big]^2}.$$

Now you have a two-dimensional integral over the triangular region $0 \le \alpha \le 1, 0 \le \beta \le 1-\alpha$.

We can write the denominator as a linear function of β:

$$D(\beta) = m^2 - \alpha\, Z - \beta\,\big(Z + \alpha\, Z_3\big).$$

Then

$$\int_0^{1-\alpha} \frac{d\beta}{[D(\beta)]^2} = \int_0^{1-\alpha} d\beta \frac{1}{\big[A + B\beta\big]^2} \quad \text{with} \quad A = m^2 - \alpha\, Z, B = Z + \alpha\, Z_3.$$

Use the elementary integration formula

$$\int \frac{dx}{(A+Bx)^2} = -\frac{1}{B}\,\frac{1}{A+Bx}\Bigg|_0^{1-\alpha} = \frac{1}{B}\Big(\frac{1}{A} - \frac{1}{A+B(1-\alpha)}\Big).$$

Hence

$$\int_0^{1-\alpha} \frac{d\beta}{[D(\beta)]^2} = \frac{1}{Z+\alpha Z_3}\left[\frac{1}{m^2-\alpha Z} - \frac{1}{m^2-\alpha Z-(1-\alpha)(Z+\alpha Z_3)}\right].$$

You now have

$$f(Z, Z_3) = \int_0^1 d\alpha \frac{1}{Z+\alpha Z_3}\left[\frac{1}{m^2-\alpha Z} - \frac{1}{m^2-(\alpha+\beta)\,Z-\alpha\beta\, Z_3}\Bigg|_{\beta=1-\alpha}\right].$$

Simplify the second denominator:

$$m^2 - \alpha Z - (1-\alpha)(Z+\alpha Z_3) = m^2 - Z - \alpha\, Z_3.$$

So

$$f(Z, Z_3) = \int_0^1 d\alpha \left[\frac{1}{(Z+\alpha Z_3)\big(m^2-\alpha Z\big)} - \frac{1}{(Z+\alpha Z_3)\big(m^2-Z-\alpha Z_3\big)}\right]. \tag{4.16}$$

Each term is now of the form $\int d\alpha/[(A+B\alpha)(C+D\alpha)]$, which you again integrate by partial fractions:

$$\frac{1}{(A+B\alpha)(C+D\alpha)} = \frac{1}{BC-AD}\Big(\frac{B}{A+B\alpha} - \frac{D}{C+D\alpha}\Big).$$

Apply this formula to each bracketed term, evaluate from $\alpha = 0$ to 1, and collect logarithms. After some algebra one arrives at the standard closed form

$$
\begin{aligned}
f(Z, Z_3) = & \frac{1}{m^2 Z_3 + Z(Z - 4m^2)} \\
& \Big[\frac{2\sqrt{Z}}{\sqrt{Z - 4m^2}} \ln\Big(\frac{2m^2 - Z + \sqrt{Z(Z - 4m^2)}}{2m^2} \Big) \\
& - \frac{Z_3 - 2Z}{\sqrt{Z_3(Z_3 - 4m^2)}} \ln\Big(\frac{2m^2 - Z_3 + \sqrt{Z_3(Z_3 - 4m^2)}}{2m^2} \Big) \Big].
\end{aligned} \tag{4.17}
$$

Clearly, we can find the singularity conditions here as

$$Z_3 \big[m^2 Z_3 + Z(Z - 4m^2) \big] = 0 \tag{4.18}$$

$$Z(Z - 4m^2) = 0 \tag{4.19}$$

$$Z_3(Z_3 - 4m^2) = 0. \tag{4.20}$$

From 4.17 one can see the explicit branch cuts and logarithms become singular exactly when the arguments under a square root or in a logarithm hit zero. Hence the singular curves are precisely those in 4.3-4.20. If one observes the physical-sheet (threshold) conditions, then one can find that the doubly coincident singularities associated with 4.3 are found to occur at:

$$\alpha = \beta = m^2/Z$$

It only lies in the integration region $0 < \alpha + \beta < 1$ if

$$Z < 2m^2.$$

Conversely, the same pinch for $Z > 2m^2$ moves into the real integration domain and produces the familiar triangular-threshold anomaly at $Z = 2m^2$. The corresponding values of (α) and (β) lie along the integration contours, and a straightforward calculation shows that they indeed pinch them. This explains the sudden appearance of the anomalous threshold on the physical sheet, first identified by [11].

Chapter 5

Conclusion

After reviewing the Polkinghorne and Screaton paper, we can now have a clear and intuitive idea to understand Landau's equation. In a broad way, it showed how integration paths in Feynman-parameter space get "pinched" when internal momenta go on-shell, creating real physical thresholds. This makes it easy to see the difference between leading singularities—when all internal lines go on-shell—and weaker ones, where only some do.

By observing the one-loop examples like triangle and box diagrams, it follows that both normal and anomalous thresholds can be written in terms of matrix determinants. The third-order vertex function then illustrates how some singularities appear only after analytic continuation to other Riemann sheets. In summary, from a physical and intuitive perspective, this paper demonstrates that it greatly simplifies the task of identifying energy scales for new particle production and complex amplitude calculations in both theoretical and experimental high-energy physics.

Appendix A

Landau singularities in square-box

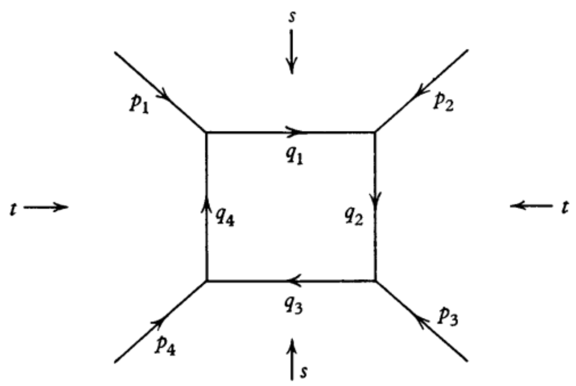

Figure A.1: The square Feynman diagram.

We are taking the one-loop square box graph case as an example. The primary objective is to identify the singularity surface. Regarding this, the calculation is similar to the triangle graph shown in Section 3.3.1. For our convention, we have:

$$p_1 + p_2 + p_3 + p_4 = 0 \tag{A.1}$$

So, the Landau equations can be written as follows:(here we keep the external and internal momentum mass same m)

- On-shell condition: By taking the internal momentum as $q_1 = k$, $q_2 = p_1 +$

k, $q_3 = p_1+p_2+k$, $q_4 = k-p_4$, we can write the on-shell condition as follows:

$$k^2 = m^2, \tag{A.2}$$

$$(p_1 + k)^2 = m^2 \Longrightarrow p_1^2 + 2\,p_1 \cdot k = 0, \tag{A.3}$$

$$(p_1 + p_2 + k)^2 = 0 \Longrightarrow (p_1 + p_2)^2 + 2\,(p_1 + p_2)\cdot k = 0, \tag{A.4}$$

$$(k - p_4)^2 = 0 \Longrightarrow p_4^2 - 2\,p_4 \cdot k = 0. \tag{A.5}$$

- Momentum-derivative condition:

$$\begin{aligned}
\sum_{i=1}^{4} \alpha_i \frac{\partial q_i}{\partial k} &= 0 \\
\Longrightarrow \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 + \alpha_4 q_4 &= 0, \\
\Longrightarrow (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\, k \;+\; \alpha_2\, p_1 + \alpha_3\,(p_1 + p_2) \;-\; \alpha_4\, p_4 &= 0, \\
\Longrightarrow k \;=\; \alpha_4\, p_4 \;-\; \alpha_2\, p_1 \;-\; \alpha_3\,(p_1 + p_2)&,
\end{aligned} \tag{A.6}$$

where, we use $\left[\sum_{i=1}^{4} \alpha_i = 1\right]$. Substituting k from A.6 to A.3, A.4 and A.5 :

$$p_1^2 + 2\,p_1\cdot(\alpha_4 p_4 - \alpha_2 p_1 + \alpha_3 (p_1 + p_2)) = 0 \Longrightarrow 2\,p_1^2\,\alpha_2 + 2\,p_1\cdot(p_1 + p_2)\,\alpha_3 - 2\,p_1\cdot p_4\,\alpha_4 = p_1^2, \tag{A.7}$$

$$2\,p_1\cdot(p_1 + p_2)\,\alpha_2 + 2\,(p_1 + p_2)^2\,\alpha_3 - 2\,p_1\cdot(p_1 + p_2)\,\alpha_4 = p_1 + p_2 \tag{A.8}$$

$$2\,p_1\cdot p_4\,\alpha_2 + 2\,(p_1 + p_2)\cdot p_4\,\alpha_3 - 2\,p_4^2\,\alpha_4 = -\,p_4^2. \tag{A.9}$$

The linear relation of α_2, α_3 and α_4 these equations gives us the 3×3-matrix, and from the determinant we can have some equations as follows, which will be needed later on to calculate our preferred solution.

$$\begin{aligned}
\Delta = 2p_1^2 &\left\{-2\,(p_1 + p_2)^2\cdot 2\,p_4^2 + 2\,(p_1 + p_2)\cdot p_4 \times 2\,(p_1 + p_2)\cdot p_4\right\} \\
-2\,p_1\cdot(p_1 + p_2)&\left\{-2\,p_1\cdot(p_1 + p_2) \times p_4^2 \;+ 2\,(p_1\cdot p_4) \times 2\,(p_1 + p_2)\cdot p_4\right\} \\
-2\,(p_1\cdot p_4)&\left\{2\,p_1\cdot(p_1 + p_2) \times 2\,(p_1 + p_2)\cdot p_4 - 2\,(p_1\cdot p_4) \times 2\,(p_1 + p_2)^2\right\} \\
&= -8\,p_1^2\,(p_1 + p_2)^2\,p_4^2 + 8\,p_1^2\left[(p_1 + p_2)\cdot p_4\right]^2 \\
+8\,(p_1\cdot(p_1 + p_2))^2\,p_4^2 &- 8\,(p_1\cdot(p_1 + p_2))\,(p_1\cdot p_4) \times 2\,(p_1\cdot(p_1 + p_2)) \\
&+8\,(p_1\cdot p_4)^2\,(p_1 + p_2)^2 \\
= -8\,p_1^2\,(p_1 + p_2)^2\,p_4^2 &+ 8\left[(p_1 + p_2)\cdot p_4\right]^2 + 8\,(p_1\cdot(p_1 + p_2))^2\,p_4^2 \\
+8\,(p_1\cdot p_4)^2\,(p_1 + p_2)^2 &- 16\,(p_1\cdot(p_1 + p_2))\,(p_1\cdot p_4)\,(p_1 + p_2)\cdot p_4.
\end{aligned} \tag{A.10}$$

For Δ_{α_2}:

$$\begin{aligned}\Delta_{\alpha_2} = p_1^2 &\Big\{-2\,(p_1+p_2)^2\times 2\,p_4^2 + 2\,p_4\cdot(p_1+p_2)\times 2\,(p_1+p_2)\cdot p_4\Big\}\\ &- 2\,p_1\cdot(p_1+p_2)\,\Big\{-(p_1+p_2)^2\times 2\,p_4^2 + p_4^2\times 2\,(p_1+p_2)\cdot p_4\Big\}\\ &- 2\,(p_1\cdot p_4)\,\Big\{(p_1+p_2)^2\times 2\,(p_1+p_2)\cdot p_4 + p_4^2\times 2\,(p_1+p_2)^2\Big\}\\ = &-4\,p_1^2\,(p_1+p_2)^2\,p_4^2 + 4\,p_1^2\,\big[(p_1+p_2)\cdot p_4\big]^2\\ &+ 4\,(p_1\cdot(p_1+p_2))^2\,p_4^2 + 4\,p_4^2\,(p_1\cdot(p_1+p_2))\times p_4\cdot(p_1+p_2)\\ &- 4\,(p_1\cdot p_4)\,(p_1+p_2)^2\times(p_1+p_2)\cdot p_4 - 4\,p_4^2\,(p_1\cdot p_4)\,(p_1+p_2)\\ = &-4\,p_1^2\,(p_1+p_2)^2\,p_4^2 + 4\,\big[(p_1+p_2)\cdot p_4\big]^2 + 4\,(p_1\cdot(p_1+p_2))^2\,p_4^2\\ &+ 4\,\big(p_1\cdot(p_1+p_2)\big)\big(p_4\cdot(p_1+p_2)\big) - 16\,(p_1\cdot(p_1+p_2))\,(p_1\cdot p_4)\,(p_1+p_2)\cdot p_4.\end{aligned} \tag{A.11}$$

For Δ_{α_3}:

$$\begin{aligned}\Delta_{\alpha_3} = &-4\,p_1^2\,p_4^2\,(p_1+p_2)^2 - 4\,p_1^2\,p_4^2\,\big(p_1\cdot p_4\big)\\ &+4\,p_1^2\,p_4^2\,\big(p_1\cdot(p_1+p_2)\big) - 4\,p_1^2\,\big(p_1\cdot p_4\big)\,\big(p_4\cdot(p_1+p_2)\big)\\ &+4\,(p_1\cdot p_4)\,\big[p_1\cdot(p_1+p_2)\big]\times p_4^2 + 4\,\big(p_1\cdot p_4\big)^2\,(p_1+p_2)^2.\end{aligned} \tag{A.12}$$

For Δ_{α_4}:

$$\begin{aligned}\Delta_{\alpha_4} = &-4\,p_1^2\,(p_1+p_2)^2\,p_4^2 - 4\,p_1^2\,(p_1+p_2)^2\;\times(p_1+p_2)\cdot p_4\\ &+4\,\big(p_1\cdot(p_1+p_2)\big)^2\,p_4^2 + 4\,p_1^2\,\big(p_1\cdot(p_1+p_2)\big)\times\big(p_1\cdot p_4\big)\\ &+4\,p_1^2\times p_1\cdot(p_1+p_2)\times\big(p_1+p_2\big)\cdot p_4 - 4\,p_1^2\,(p_1+p_2)\times p_1\cdot p_2.\end{aligned}$$

Now, from here we can find the α's value:

$$\begin{aligned}\alpha_2 &= \frac{\Delta_{\alpha_2}}{\Delta},\\ \alpha_3 &= \frac{\Delta_{\alpha_3}}{\Delta},\\ \alpha_4 &= \frac{\Delta_{\alpha_4}}{\Delta}\end{aligned} \tag{A.13}$$

Putting this values in the on-shell condition, we get:

$$\begin{aligned}&\{\alpha_4 p_4 - \alpha_2 p_1 - \alpha_3(p_1+p_2)\}^2 = m^2,\\ \implies\quad &\alpha_2^2\,p_1^2 + \alpha_2\,p_1^2 + \alpha_3^2\,(p_1+p_2)^2 - 2\,\alpha_2\,\alpha_3\,p_1\cdot p_4\\ &- \alpha_3\,\alpha_4\,p_1\cdot(p_1+p_2) + 2\,\alpha_2\,\alpha_3\,p_1\cdot(p_1+p_2) = m^2.\end{aligned} \tag{A.14}$$

So, finally, we get the Landau surface where the singularities arise for the one-loop square box case.

Appendix B

Feynman Parameterization

We start with the Feynman integral 4.1 with defining n-internal linesand m-loops, n-nonnegative Feynman parameter $\alpha_1, ..., \alpha_n$, and a delta function $\delta(\sum \alpha_n - 1)$:

$$\lim_{\varepsilon \to 0^+} \int_0^\infty d\alpha_1 \cdots d\alpha_n \delta\big(\alpha_1 + ... + \alpha_n - 1\big) \int d^4k_1 \cdots d^4k_m \frac{1}{[F(\alpha_i, k_j, p_k) + i\varepsilon]^n} \tag{B.1}$$

Where,

$$F(\alpha_i; k_j; p_k) = \sum_{i=1}^{n} \alpha_i \big(q_i^2 - m_i^2\big) \tag{B.2}$$

We can write the **Schwinger**-perameter ν-via well known identity:

$$\frac{1}{(F + i\varepsilon)^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\nu \, \nu^{(n-1)} \exp(-\nu(F + i\varepsilon), \quad n > 0 \tag{B.3}$$

Inserting this equation into B.1, we can have the Schwinger-representation :

$$I = \frac{1}{\Gamma(n)} \lim_{\varepsilon \to 0^+} \int_0^1 \prod_{i=1}^{n} d\alpha_i \, \delta \left(\sum_{i=1}^{n} \alpha_i - 1 \right) \int_0^\infty d\nu \, \nu^{n-1} \, \exp\left(-\, i \, \varepsilon \, \nu\right) \int \prod_{r=1}^{m} d^4k_r \, \exp(-\, \nu F(\alpha_i, k_r)) \tag{B.4}$$

Here, we can drop $e^{-i\varepsilon\nu}$ as $\varepsilon \to 0^+, e^{-i\varepsilon\nu} \approx 1$.
We know in B.2, q_i is a linear combination of the loop momenta (k_r) and the external momenta (p_s). One writes:

$$q_i^\mu = \sum_{r=1}^{m} \eta_{ir} \, k_r^\mu \; + \; \sum_s \hat{\eta}_{is} \, p_s^\mu, \tag{B.5}$$

We can write the quadratic equation as:

$$\boxed{\sum_{i=1}^{n} \alpha_i \, q_i^2 = k^T \, M(\alpha) \, k + 2 \, k^T \, Q(\alpha, p) + J(\alpha, p),} \tag{B.6}$$

with the matrix in a form of:

$$M_{(r\mu)\,(s\nu)} = \sum_i \alpha_i\, \eta_{ir}\, \eta_{is}\, g_{\mu\nu}, \quad Q_{(r\mu)} = \sum_i \alpha_i\, \eta_{ir} \left(\sum_t \hat{\eta}_{it} p_t\right)_\mu, \quad J = \sum_i \alpha_i \left(\sum_s \hat{\eta}_{is} p_s\right)^2. \tag{B.7}$$

Shifting the loop momentum as:

$$k \;\rightarrow\; k - M^{-1}Q,$$

can give:

$$F(\alpha, k, p) = K^T\, M(\alpha)\, K - Q(\alpha, p)^T\, M(\alpha)\, Q(\alpha, p) + J(\alpha, p) \tag{B.8}$$

From this we can also define the function $F'(\alpha, p)$ by elimininating the k-dependence part:

$$F'(\alpha) \equiv J - Q^T M^{-1} Q. \tag{B.9}$$

Now, if we look at the integral, we can observe that each of the m-loop integrals is now a standard Gaussian in 4 dimensions.So, applying the Gaussian integral:

$$\int \prod_{r=1}^{m} d^4 k_r\, \frac{1}{(k^T\, M\, k\; + F'(\alpha, p) + i\varepsilon)^n} = \frac{\pi^{2m}}{(det\, M(\alpha))^{1/2}}\, \frac{1}{(F'(\alpha, p) + i\varepsilon)^{n-2m}} \tag{B.10}$$

Rescale the Feynman parameters to extract the overall scale ν.Define:

$$\nu_i \equiv \nu\, \alpha_i, \quad \sum_{i=1}^{n} \nu_i = \nu, \quad \sum_i \alpha_i = 1. \tag{B.11}$$

Putting all pieces together:

$$I = \pi^{2m} \int_0^\infty \prod_{i=1}^{n} d\nu_i\, \frac{e^{-\sum_{i=1}^{n} \nu_i\, m_i^2}}{\sqrt{\det M(\nu)}} \delta\Big(\sum_i \nu_i - \nu\Big)\, \frac{1}{[F'(\nu, p) + i\varepsilon]^{n-2m}} \tag{B.12}$$

Finally, we can typically trades back ν_i for the $\alpha_i$and the scale ν, as in the previous step, obtaining precisely the standard Feynman-parameter form:

$$\boxed{I = \lim_{\varepsilon \to 0^+} \int_0^1 d\alpha_1 \cdots d\alpha_n\, \delta\left(\sum_i \alpha_i - 1\right) \frac{\varphi(\alpha)}{\left[F'(\alpha, p) + i\varepsilon\right]^{n-2m}}} \tag{B.13}$$

This is the exact equation that we use in 4.3

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Approval

The internship report titled "**A Guide to Analytic Structure of Feynman Amplitudes and Landau Singularities**" submitted by **Noshin Ferdous Shamma**, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfilment of the requirements of the internship program.

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