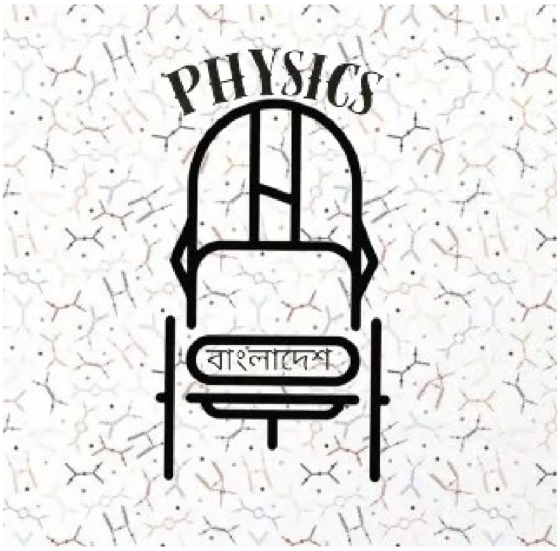
Simulating an Expanding Universe



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Abstract

We present a numerical simulation of an expanding universe based on the model of modern cosmology given by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. We utilize the Friedmann equations to construct a system of differential equations and implement a high precision integration function to track the evolution of the scale factor and density parameters from the Big Bang to the current day. Our simulation correctly identifies the different cosmic epochs along with their transitions through computing the evolution of the density parameters. We calculate important cosmological factors including the Hubble Parameter, lookback time, and redshift as functions of the scale factor $a(t)$. Our results yield a present day age of the universe of 13.80 Gyr and a Hubble constant of 67.66 km/s/Mpc which match excellently with observed results from Planck 2018. Matter-Radiation equality in our simulation occurs at $z_{eq} = 3387$, matching theoretical predictions. Our work demonstrates the excellent predictive power of the Friedmann equations and validates their ability to replicate the conditions of modern cosmology through numerical integration.

Contents

1 Introduction

The evolution of our Universe is governed by the interplay of matter, energy, and the expansion of space-time. It spans a history of 13.8 Billion years starting from the Big Bang. One of the core objectives of modern Cosmology is to understand this evolution and structure of the Universe. The Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which coupled with Einstein's field equations of General Relativity, allows us to map out the dynamics between the scale factor and the changes of the density parameters that gave rise to different epochs of our universe.

1.1 Chronology of the Universe

The history of our universe can be classified into different epochs constituting different dominant physical processes and energy densities:

1. ***The Planck Epoch ($t < 10^{-43}$ s):*** The Earliest period right after the Big Bang when cosmology and physics is thought to have been dominated by quantum effects of gravity. Our current models break down at this scale.

2. ***Inflationary Epoch($10^{-36} \leq t \leq 10^{-32}$ s):*** This was a period of extremely rapid expansion which is believed to have smoothed out initial inhomogeneities and generated the initial density fluctuations that later gave rise to all structure

3. ***Radiation Domination ($t < 47,000$ years):*** Following inflation, the universe was filled by a dense plasma of relativistic photons and neutrinos which dominated over matter and dark energy. Atoms could not form during this period due to the domination of energetic photons

4. ***Matter Domination ($47,000$ years $< t \leq 9.8$ billion years):*** As the expansion of the universe continued and the temperature of the universe continued to drop, the universe started to become dominated by non-relativistic matter. This period saw the emergence of the first atoms through the combination of electrons with atomic nuclei. The Cosmic Microwave Background was formed during this era and gravitational collapse led to the formation of the first Stars and Galaxies.

5. ***Dark Energy Domination ($t > 9.8$ billion years):*** As the universe continued to expand, the density of Dark Energy surpassed that of matter and radiation leading to the acceleration of the expansion of our universe. This is due to the negative pressure associated with Dark Energy (modeled as the cosmological constant). We are currently in this era of the universe

1.2 Objectives

This report aims to present a numerical simulation of our expanding universe using the Friedmann equations and initial conditions. The objectives of our simulation are as follows:

1. To solve the Friedmann Equations and track the evolution of the density parameters through the history of our universe
2. To identify and characterize the transitions between different cosmological eras (radiation-dominated, matter-dominated, and dark energy-dominated)
3. To compute the evolution of observable quantities such as the scale factor, redshift, and Hubble parameter, comparing our results with established cosmological data.

1.3 Units and Convention

Throughout this report, we adopt the fundamental constants are set to unity:

$$c = \hbar = k_{\mathrm{B}} = 1$$

In this system, energy, mass, momentum, and temperature all have the same unit (eV). Consequently, length and time are equivalent and have units of inverse energy. Below, we present the theory and relevant code application of our numerical simulation.

2 The Cosmological Principle

The cosmological principle is at the heart of our modern formulation of cosmology. It is the important fact we presuppose to be true to build our cosmological models on. The cosmological principle states that the universe is both homogeneous and isotropic on large scales. Specifically, on scales of $\sim \mathcal{O}(\mathbf{10^2})$ Mpc or larger. That is, when averaged over sufficiently large volumes, the universe appears statistically uniform in its properties.

2.1 Homogeneity

Homogeneity refers to the spatial uniformity of the universe. That is, the concept that there is no preferred location in space. Thus, at large enough scales, space looks the same from all points as shown in 1. Mathematically, this means that over a sufficiently large region, the statistical properties of the universe such as matter density, expansion rate,etc are independent of location. And so, a homogeneous universe shows translational symmetry.

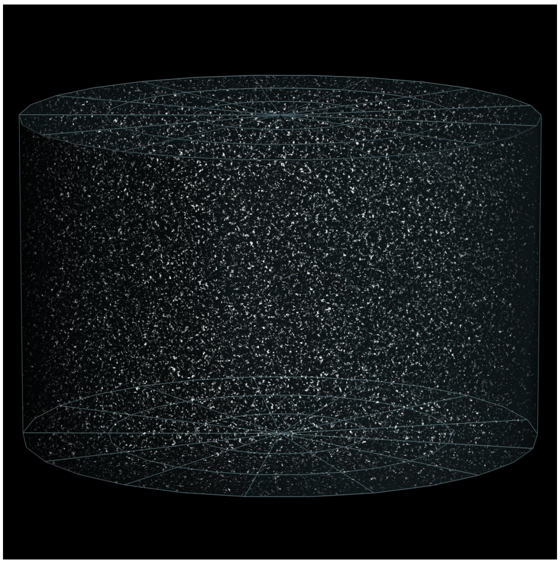

Figure 1: Surveys of our observable universe confirms homogeneity at large scales. Credit:ESO/Frédéric Michel

2.2 Isotropy

Isotropy refers to the directional uniformity of space. That is, the concept that there is no preferred direction in space. An Isotropic universe looks the same in all directions and thus, preserves rotational symmetry. Observationally, Isotropy is observed through the distribution of galaxies in large scales which we can see in 2 as well as in the miniscule temperature fluctuation of the CMB.

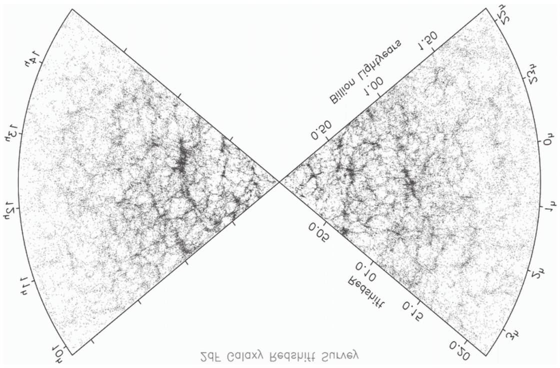

Figure 2: Isotropy of Galaxy distribution as observed in the redshift survey conducted at the Anglo-Australian Observatory

2.3 On Modeling the Universe

The Cosmological Principle is the most important presupposition when modeling the universe[1]. As the universe is both homogeneous and isotropic at sufficiently large scales, we are able to model our universe as a perfect fluid which is characterized by having no shear stress or viscosity. This assumption allows us to massively simplify the model of our universe by allowing us to use the Stress-Energy Tensor for a perfect fluid. Applying this assumption on the Einstein field equation allows us to solve for the Friedmann equations[2]. It must be clarified that the universe is only homogeneous and isotropic on sufficiently large scales of $\sim \mathcal{O}(\mathbf{10^2})$ Mpc or larger. On smaller scales, the universe is inhomogeneous and anisotropic due to the structure of Galaxies and Stars.

3 Friedmann Equations

The Friedmann Equations give us the expansion of a universe following the cosmological principle. Using the Friedmann equations we can model the rate of expansion along with the change of different energy densities with time. This also allows us to model the different epochs of the universe along with the interaction of different types of energies to the rate of expansion. To derive the Friedmann equations, we can use the cosmological principle on the Einstein field equation. Alternatively, the Friedmann equations can be derived purely from Newtonian mechanics using certain assumptions. Here, we will derive the Friedmann equations using Newtonian mechanics while showing the important assumptions and factors of the GR derivation.

3.1 Friedmann-Lemaître-Robertson-Walker metric

To solve the Einstein field equation for a homogeneous and isotropic universe, we need a metric that includes all the aspects of the cosmological principle. The metric that satisfies this condition is the Friedmann-Lemaître-Robertson-Walker metric. The construction of the FLRW metric is guided by the symmetries imposed by the cosmological principle. Homogeneity requires the metric to be invariant under spatial translations while Isotropy requires the metric to be invariant under spatial rotations. This severely constraints the possible forms of the spacetime metric. The metric satisfying the conditions is given by the FLRW metric.

The Friedmann-Lemaître-Robertson-Walker metric is given by:

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1 - r^2/k^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) \tag{1}$$

where $a(t)$ is the scale factor of expansion, k is the spatial curvature and (θ, ϕ) are polar and azimuthal angles in a spherical coordinate system

The space of this metric is characterized by the constant spatial curvature k and falls into three distinct classes [3]:

1. **Positive Curvature ($k > 0$):** This Universe has positive curvature and has the geometry of a three sphere. It is characterized by a closed geometry. Here, k can be normalized to $k = 1$. An universe with positive curvature has supercritical density. In such a universe, gravity eventually overcomes the negative pressure exerted by Dark Energy. Thus, the scale factor reaches a maximum value and the universe starts contracting leading to the universe collapsing in a "Big Crunch" which might lead to another Big Bang.. Such a universe might lead to cycles of expansions and contractions

2. **Zero Curvature ($k = 0$):** This Universe has no curvature and is flat. It has the geometry of 3D Euclidean space with infinite volume. In such a universe, the critical density is enough to slow the expansion of the universe but not enough to halt it. In such a universe, the expansion slows as time advances but expansion never stops. This leads to a "Heat Death" ending where the stars and even black holes eventually die out given enough time.

3. **Negative Curvature ($k < 0$):** This Universe has negative curvature and has the geometry of a hyperbolic three dimensional space with infinite volume. This universe has an open geometry and resembles a saddle. k here can be normalized to $k = -1$. In such a universe, the rate of expansion increases continuously leading to ever runaway expansion and a "Big Rip" ending where galaxies and even stars are thrown away from each other by the expansion of spacetime.

An alternative form of the FLRW metric uses radial coordinate χ defined by $r = S_k(\chi)$. Through observational evidence we find our universe to be geometrically flat with $k = 0$

3.2 Scale factor $a(t)$

The scale factor denoted by $a(t)$ is a dimensionless unit that describes how the universe is changing with respect to it's size at a previous time. That is, it describes the rate of change of the rate of change of the expansion of the universe. Naturally, the scale factor is a changing parameter and we can use it to determine the rate of expansion of the universe at a certain point in the past or future with respect to another point in time (like the present). The Friedmann equation allows us to express the scale factor as a function of energy density which allows us the find the rate of the expansion of the universe through analyzing its energy content.

3.3 Newtonian Derivation of Friedmann Equations

We consider a sphere enclosing a mass M with radius $R(t)$, illustrated in Fig. (3). The change of the sphere radius is given by the gravitational acceleration of the outer edge of the sphere. So, using Newton's law:

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} \tag{2}$$

$$\left(\frac{dR}{dt}\right)\frac{d^2R}{dt^2} = -\frac{GM}{R^2}\left(\frac{dR}{dt}\right) \tag{3}$$

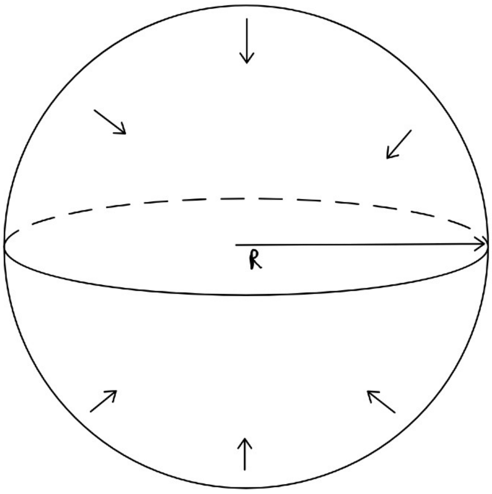

Figure 3: This figure illustrates a sphere enclosing a mass M experiencing inward gravitational acceleration.

We take $\frac{dR}{dt} = V$ and write:

$$\int (VdV) = -GM \int \left(\frac{dR}{R^2}\right) \tag{4}$$

$$\frac{1}{2}V^2 = \frac{GM}{R} + U \tag{5}$$

$$\frac{1}{2}\left(\frac{dR}{dr}\right)^2 = \frac{GM}{R} + U \tag{6}$$

where U is a constant of integration and is the energy per unit mass at the outer surface of the expanding sphere. From the equation we can see that if $U > 0$, (positive total energy) the sphere keeps expanding infinitely. On the other hand if $U < 0$ (negative total energy), the sphere eventually stops expanding and the radius collapses back to 0 [2, 4].

We can write the radius as

$$R(t) = a(t)r \tag{7}$$

where $a(t)$ is the scale factor and r is the radius at a certain point in time. So the scale factor causes the radius to change.
Now, the mass inside the sphere is given by

$$M = \frac{4\pi}{3}\rho R^3 \tag{8}$$

Putting these into our equation, we find:

$$\frac{1}{2}r^2\dot{a}^2(t) = \frac{4\pi}{3}Gr^2\rho(t)a^2(t) + U \tag{9}$$

Dividing both sides by $r^2a^2/2$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho(t)G}{3} + \frac{2U}{r^2(t)a^2(t)} \tag{10}$$

$$= \frac{8\pi\rho(t)G}{3} - \frac{\kappa c^2}{r^2(t)a^2(t)} \tag{11}$$

, we find: where $\kappa c^2 = -\frac{2U}{r^2(t)}$.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{2U}{r^2a^2} \tag{12}$$

$$= \frac{8\pi G}{3}\rho - \frac{\kappa c^2}{r^2a^2} \tag{13}$$

This is the first Friedmann equation [4]. From this, we define the Hubble parameter

$$H = \frac{\dot{a}}{a} \tag{14}$$

Now, we can define critical density

$$\rho_c = \frac{3H_0^2}{8\pi G} \tag{15}$$

To properly describe the evolution of the universe, we need an equation to describe the evolution of energy density along with the scale factor given by the first Friedmann equation. We know from Thermodynamics:

$$dE + pdV = Tds$$

For an expanding volume of unit fixed radius, and using $E = mc^2$, we find:

$$(E) = \frac{4\pi}{3}\rho a^3 c^2 \tag{16}$$

$$\left(\frac{dE}{dt}\right) = 4\pi a^2 \rho c^2 \frac{da}{dt} + \frac{4\pi}{3} a^3 c^2 \frac{d\rho}{dt} \tag{17}$$

where $\frac{dV}{dt} = 4\pi a^2 \frac{da}{dt}$. Assuming reversible expansion ($dS = 0$), we find:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + \frac{p}{c^2}) = 0 \tag{18}$$

This is known as the fluid equation. Now, differentiating the first Friedmann equation w.r.t time:

$$2\frac{\dot{a}}{a}\frac{a\ddot{a} - \dot{a}^2}{a^2} = \frac{8\pi\dot{\rho}(t)G}{3} + 2\frac{\kappa c^2 \dot{a}}{a^3} \tag{19}$$

Subbing in the value of $\dot{\rho}$ from the fluid equation:

$$\frac{\ddot{a}}{a} - (\frac{\dot{a}}{a})^2 = -4\pi G(\rho + \frac{p}{c^2}) + \frac{\kappa c^2}{a^2} \tag{20}$$

Again, using the first Friedmann equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + \frac{3p}{c^2}) \tag{21}$$

This is the second Friedmann equation, also known as the acceleration equation[5].

4 Density Parameters

The evolution of the universe is determined by its energy content which governs the expansion rate and even the ultimate fate of our universe. This energy content is characterized by density parameters Ω which also help to divide the energy content into different components. The density parameter for a component i is defined as the ratio between its energy density $\rho_i(t)$ and critical density $\rho_c(t)$ given by

$$\Omega_i(t) = \frac{\rho_i(t)}{\rho_c(t)} \tag{22}$$

where the critical density is the energy density required for the universe to be spatially flat $k = 0$. The parameters evolve with the scale factor as

$$\Omega_i(a) = \frac{\rho_{i,0} a^{-n_i}}{\rho_c(a)} = \frac{\Omega_{i,0} a^{-n_i}}{E^2(a)} \tag{23}$$

where n_i is the scaling exponent (4 for radiation, 3 for matter, 2 for curvature, 0 for dark energy) and $E(a)$ is the dimensionless Hubble parameter.

The total density parameter is the sum of all components:

$$\Omega_{\text{total}}(t) = \Omega_m(t) + \Omega_r(t) + \Omega_\Lambda + \Omega_k \tag{24}$$

where the subscripts denote Matter, Radiation, Dark Energy, and Spatial Curvature respectively.[2]

4.1 Curvature Parameter

The spatial curvature of our universe also contributes to the Friedmann equation. We define the curvature parameter

$$\Omega_k = -\frac{k}{a^2 H^2} \tag{25}$$

From the first Friedmann equation, we can write:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} = \frac{8\pi G}{3}\rho_c \left(\frac{\rho}{\rho_c}\right) - \frac{k}{a^2} \tag{26}$$

Dividing by H^2 and rearranging the terms, we find:

$$\Omega_{\text{total}} = 1 + \Omega_k \tag{27}$$

And so, we find :

- If $\Omega_{\text{total}} = 1$, $\Omega_k = 0$ and $k = 0$ (The universe is flat)
- If $\Omega_{\text{total}} > 1$, $\Omega_k < 0$ and $k > 0$ (The universe is closed)
- If $\Omega_{\text{total}} <$, $\Omega_k > 0$ and $k < 0$ (The universe is open)

From observational evidence, we know that the universe is spatially flat to a high precision $|\Omega_k| < 0.005$. Thus, in our simulation we take $\Omega_{\text{total}} = 1$ and $k = 0$

4.2 Matter Density Parameter

The matter density parameter(Ω_m) contains both baryonic matter and dark matter. It exerts negligible pressure ($\rho \approx 0$) and scales as a^{-3} with the scale factor

4.3 Radiation Density Parameter

The radiation density parameter(Ω_r) contains relativistic photons and neutrinos. The energy density of the parameter scales as a^{-4} with the scale factor and thus dilutes the fastest with the expansion of spacetime

4.4 Dark Energy Density Parameter

The dark energy density, characterized by the cosmological constant (Ω_Λ) indicates the dark energy density of our universe. This parameter exerts a negative pressure on our universe $w = -1$ and increases the expansion rate of the universe. Unlike other parameters, Ω_Λ does not dilute with time and remains constant. This leads to increasing density of dark energy and an increase in the rate of expansion of spacetime

4.5 Friedmann equation in terms of Density Parameters

We can express the Friedmann equation in terms of density parameters using $H = H_0 E(a)$ where $E(a)$ is the dimensionless Hubble parameter dependent on the scale factor.

$$E(a) = \frac{H(a)}{H_0} = \sqrt{\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{k,0}a^{-2} + \Omega_{\Lambda,0}} \tag{28}$$

5 Distance Measures

Due to the expansion of the universe, along with the limitations of Earth based observations, different measures of distance is used in cosmology and astronomy. Some of these are:

5.1 Comoving Distance

Due to the expansion of the universe, the space and distance between any two objects is constantly increasing. Even if the objects are at rest with each other, the expansion of space between them leads to increase in distance between them. The further apart they are, the faster they move apart due to the expansion of space. However, it is necessary to measure the distance between objects without the expansion of space to find properties like the object's velocity. For this, we can use comoving distance. Comoving distance is the distance between objects while accounting for the expansion of space. The expansion of space is subtracted from distance calculation and a snapshot of the object's distance at a specific time is used as a reference[3]. Comoving distance is given by

$$\chi = \int_{t_o}^{t} c\frac{dt'}{a(t')} \tag{29}$$

5.2 Proper Distance

This is the regular notion of distance between any two objects at a specific time. This is a changing quantity even if the objects remain at rest w.r.t each other due to the expansion of space. Proper distance is found by using the scale factor with the comoving distance to account for expansion. So:

$$d(t) = a(t)\chi \tag{30}$$

5.3 Angular Diameter Distance

The angular diameter distance is the measurement by which we understand an object's distance by relating its physical size with its observed angular size. That is, if we know an object's physical size and measure the apparent size from our viewpoint as shown in, we can relate it using

$$D_A = \frac{s}{\theta} \tag{31}$$

Where D_A is the distance of the object from our viewpoint. s is the size of the object and θ is the measured angle.

5.4 Redshift

Redshift is the phenomenon of light increasing in wavelength and thus, taking a more reddish hue due to the loss of energy. Unlike doppler redshift caused by the light source travelling away from the observer, cosmological redshift differs in that it is caused due the expansion of the universe stretching out light waves. As a light wave is emitted from a source with a wavelength λ_e, the emitted light must travel through space that is constantly expanding. As space expands, the wavelength of the travelling light wave is lengthened as the light loses energy to the expansion of space. Thus, the observer finds the light wave to be of a longer wavelength λ_o and the light is redshifted. The further the observer is from the source, the longer the travel time of the light from the reference of the observer and thus, the more redshifted the light is. Thus, by calculating redshift, we can find the distance of the source object from the observation point using

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e} \tag{32}$$

. We can express the Friedmann equation dependent on redshift as [3]:

$$E(z) = \sqrt{\Omega_{\Lambda,0} + \Omega_{k,0}(1+z)^2 + \Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4} \tag{33}$$

5.5 Lookback Time

Lookback time is the time difference between the age of the universe at the moment of emission of light from a source and the age of the universe when it reaches the observer. That is,it is the time it takes for light to travel from the source to the observer. Thus, if it takes 8 minutes for light from the Sun to reach the Earth, the lookback time of the Sun from an observer at the Earth is 8 minutes. Lookback time can be used to determine the age of objects and even the age of the universe by using the Cosmic Microwave Background emitted during the Big Bang as a source. However, for distant objects we must also factor in the expansion of the universe when calculating lookback time as light takes longer to reach an observer in an expanding universe compared to a static universe. From our above discussion on redshift we understand that redshift is linked to lookback time as the higher the redshift of an object, the larger the lookback time. Lookback time is derived using the Friedmann equations and is calculated using

$$t_l(z) = t_H \int_0^z \frac{dz'}{(1+z')E(z')} \tag{34}$$

where

$$E(z) = \sqrt{\Omega_m(1+z)^3 + (1 - \Omega_M - \Omega_\Lambda)(1+z)^2 + \Omega_\Lambda} \tag{35}$$

with Ω_M representing current density parameters and Ω_Λ representing the cosmological constant. We can also write this in terms of the scale factor as:

$$t_{\text{lookback}}(a) = \frac{1}{H_0} \int_a^1 \frac{da'}{a'E(a')} \tag{36}$$

6 Numerical Implementation

Having consolidated our theory, we can proceed to implement said theory to simulate the evolution of our universe. We do so through implementing numerical solution to the Friedmann equations using python with the help of the following scientific computing libraries: NumPy for numerical operations, SciPy for integration and interpolation, and Matplotlib for visualization.

We adopt the following cosmological parameters based on recent Planck observations1:

```
# Constants
H0 = 67.66  # Hubble Constant [km/s/Mpc]
Om0 = 0.311  # Matter density parameter
Ode0 = 0.689  # Dark energy parameter
Or0 = 9.18e-5  # Radiation density parameter
Ok0 = 1 - (Om0 + Or0 + Ode0)  # Curvature density parameter
H0_Gyr = H0 * 1.02275e-3  # H0 in Gyr^-1
H0_inverse = 977.8 / H0  # 1/H0 in Gyr
```

Listing 1: Cosmological Constants and Parameters

These parameters satisfy the flatness condition $\Omega_{\text{total}} \approx 1$

6.1 Solving the Friedmann Equation

We can rewrite the Friedmann equation as a first order ODE for the scale factor:

$$\frac{da}{dt} = H_0 * a(t) * E(a) \tag{37}$$

where $E(a)$ is the dimensionless Hubble parameter given by equation 28.

6.1.1 Implementation of the Hubble Parameter

We first implement the dimensionless Hubble parameter as a function of the scale factor as shown in 2:

```python
def hubble_parameter(a: float | np.ndarray) -> float | np.ndarray:
    """
    Calculate the Hubble parameter in terms of the
    dimensionless energy parameters.

    Args: a - the scale factor of expansion
    Returns: E(a) - dimensionless Hubble parameter
    """
    return np.sqrt(Or0 * a**-4 + Om0 * a**-3 +
                   Ok0 * a**-2 + Ode0)
```

Listing 2: Hubble Parameter Function

This function implements equation 28 and computes

$$E(a) = \sqrt{\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{k,0}a^{-2} + \Omega_{\Lambda,0}} \tag{38}$$

6.1.2 Friedmann ODE Formulation

We write the Friedmann equation as an ODE function for later numerical integration using 3:

```python
def friedmann_ode(t: float, a: float) -> float:
    """
    Friedmann ODE function for scale factor evolution

    Args:
        t: Time [Gyr]
        a: Scale factor
    Returns:
        da/dt - time derivative of scale factor
    """
    return hubble_parameter(a) * H0_Gyr * a
```

Listing 3: Friedmann ODE Function

6.1.3 Numerical Integration

We solve the Friedmann ODE function we defined using SciPy's solve-ivp function with the DOP853 method (Dormand-Prince 8th order Runge-Kutta) that allow us to account for changing steps at different epochs as shown in 4:

```
# Initial conditions
t_0 = 1e-5  # Initial time [Gyr]
a_0 = 1e-10  # Initial scale factor
t_span = (t_0, 100)  # Simulation time range [Gyr]

# Solve Friedmann ODE
solution = solve_ivp(
    friedmann_ode,
    t_span,
    [a_0],
    t_eval=np.logspace(np.log10(t_0), np.log10(100), 10000)
        ,
    method='DOP853',  # Higher order method
    rtol=1e-10,  # Relative tolerance
    atol=1e-15   # Absolute tolerance
)

t_sol = solution.t  # Time array
a_sol = solution.y[0]  # Scale factor array
```

Listing 4: Numerical Solution of Friedmann Equation

Here,we implement logarithmic spacing in time to show the early universe's rapid evolution while also showing the evolution at larger time scales at later times of the universe. The higher-order integration method ensures accuracy of the scale factor through different epochs.

6.2 Computing Density Parameter Evolution

We now compute the evolution of the density parameters with time which evolve according to their respective scaling and the Hubble parameter 5:

```python
def density_evolution(a: np.ndarray) -> np.ndarray:
    """
    Calculate evolution of density parameters
    with scale factor

    Args: a - scale factor array
    Returns: (Omega_r, Omega_m, Omega_k, Omega_de)
    """
    a_safe = np.maximum(a, 1e-20)  # Avoid singularities
    E_a = hubble_parameter(a_safe)

    Omega_r = Or0 * a_safe**-4 / E_a**2
    Omega_m = Om0 * a_safe**-3 / E_a**2
    Omega_k = Ok0 * a_safe**-2 / E_a**2
    Omega_de = Ode0 / E_a**2

    return Omega_r, Omega_m, Omega_k, Omega_de
```

Listing 5: Density Parameter Evolution

This implementation follows directly from Equation 23

6.3 Lookback Time Calculation

We calculate the evolution of Lookback time with the scale factor. We compute the evolution by integrating Equation 36 as we show in 6:

```python
def lookback_time(a: np.ndarray) -> np.ndarray:
    """
    Calculate lookback time as a function of scale factor

    Args: a - scale factor array
    Returns: t_lookback - lookback time array [Gyr]
    """
    t = np.zeros(len(a))

    for i in range(len(a)):
        if a[i] >= 1:
            t[i] = 0
            continue

        integral, _ = quad(
            lambda x: 1 / (x * hubble_parameter(x)),
            a[i], 1,
            epsabs=1e-8, epsrel=1e-8, limit=500
        )
        t[i] = integral * H0_inverse

    return t
```

Listing 6: Lookback Time Integration

This numerical integration uses adaptive quadrature to handle the integrand's behavior across different cosmological epochs.

6.4 Identifying Cosmic Epochs

We now identify which component dominates at each point in the simulation, thus allowing us to identify different Cosmic Epochs. We do so by defining a function that allows us to identify epochs based on largest density parameter defined in 7:

```python
def identify_dominance_regions(t_vals, a_vals):
    """
    Identify radiation, matter, and dark energy
    dominated epochs
    """
    Omega_r, Omega_m, _, Omega_de = density_evolution(a_vals)

    regions = []
    for i in range(len(a_vals)):
        if Omega_r[i] > Omega_m[i] and Omega_r[i] > Omega_de[i]:
            regions.append(’radiation’)
        elif Omega_m[i] > Omega_r[i] and Omega_m[i] > Omega_de[i]:
            regions.append(’matter’)
        else:
            regions.append(’lambda’)

    return regions
```

Listing 7: Epoch Identification

This function allows us to automatically identify transitions between cosmological epochs

6.5 Finding Matter-Radiation Equality

Matter-radiation equality occurs when $\Omega_m = \Omega_r$. We can compute this both analytically and numerically through 8:

```
# Analytical calculation
a_eq_analytical = Or0 / Om0
z_eq_analytical = 1/a_eq_analytical - 1

# Numerical calculation from simulation
ratio = Omega_r / Omega_m
eq_index = np.argmin(np.abs(ratio - 1))
a_eq_numerical = a_sol[eq_index]
z_eq_numerical = 1/a_eq_numerical - 1
```

Listing 8: Matter-Radiation Equality

7 Discussion of Results

7.1 Scale Factor Evolution

Here, Figure 4 shows the evolution of the scale factor $a(t)$ from the early universe to our present day. Our plot is divided into three distinct regions corresponding to Radiation dominated(red), Matter dominated(blue), and Dark Energy dominated(green) epochs. In the radiation-dominated era ($t < 47,000$ years), the scale factor grows as $a(t) \propto t^{1/2}$, this is characteristic of a universe where energy density of relativistic particles dominate over other energy densities. The transition to matter domination occurs at $z \approx 3400$, after which the scale factor evolves as $a(t) \propto t^{2/3}$ until approximately $t \approx 10.3$ Gyr.

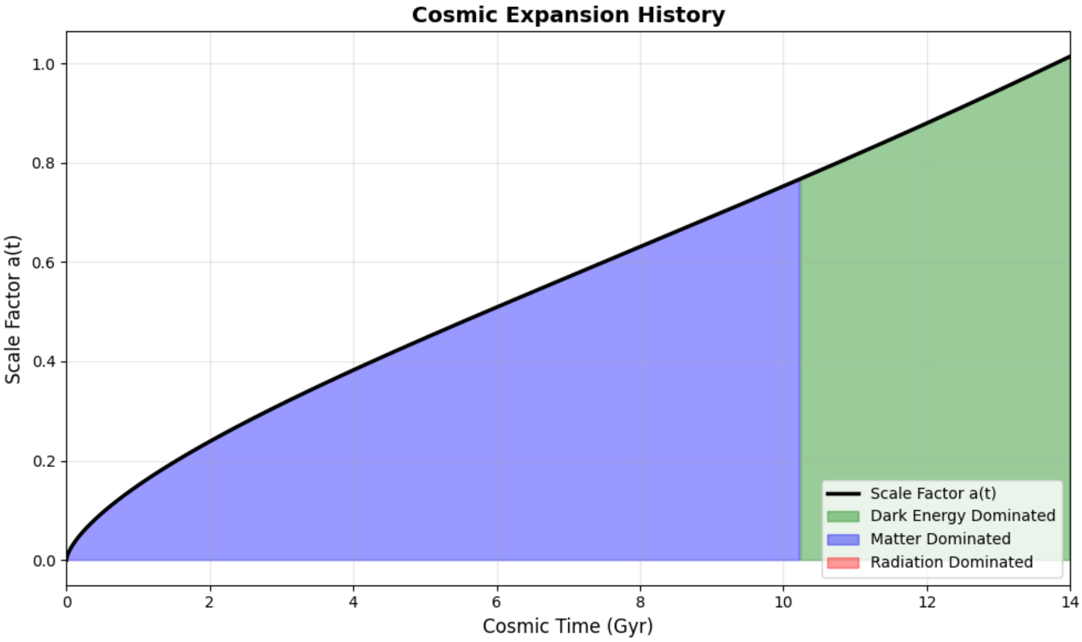

Figure 4: Evolution of the Scale Factor

At later times, Dark Energy begins to dominate over other energy densities leading to accelerated expansion. Here the scale factor approaches exponential growth: $a(t) \propto e^{Ht}$

7.2 Hubble Parameter

The Hubble parameter $H(a)$ shown in 5 decreases with our increasing scale factor. This is due to the overall diluting of energy densities. At small $a(t)$, radiation dominance causes $H \propto a^{-2}$. During matter domination, $H \propto a^{-3/2}$. As Dark Energy dominantes, H approaches a constant value $H_\infty = H_0\sqrt{\Omega_{\Lambda,0}} \approx 56$ km/s/Mpc. Our numerical results give us a present-day Hubble constant of $H_0 = 67.66$ km/s/Mpc, which is consistent with Planck 2018 measurements [6].

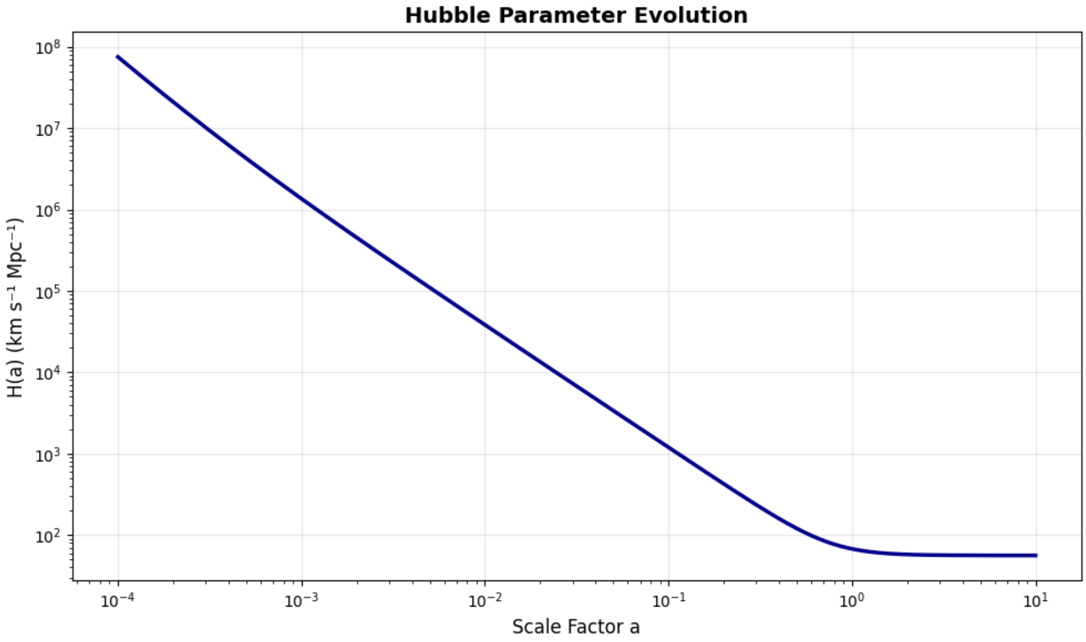

Figure 5: Hubble Parameter Evolution

7.3 Density Parameter Evolution

Figure 6 shows the evolution of our universe's density parameters as functions of scale factor $a(t)$. The different scaling behaviors are visible in the plot:

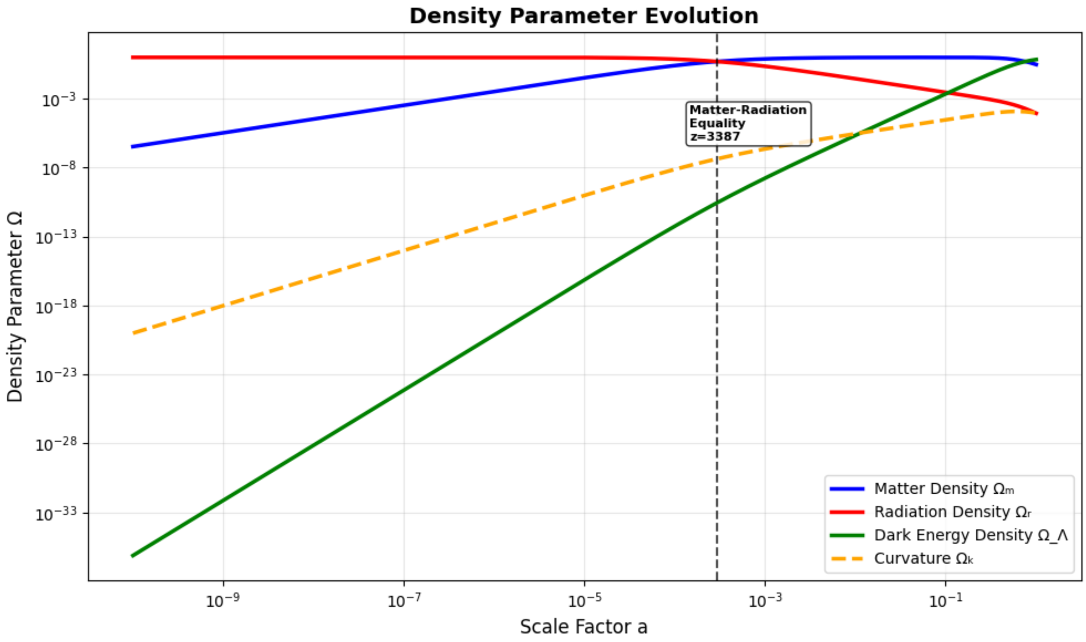

Figure 6: Density Parameter evolution

- **Radiation** ($\Omega_r \propto a^{-4}$): Dominates at $a < 3 \times 10^{-4}$ (corresponding to $z > 3400$)
- **Matter** ($\Omega_m \propto a^{-3}$): Dominates during $3 \times 10^{-4} < a < 0.75$ ($0.3 < z < 3400$)
- **Dark Energy** (Ω_Λ constant): Dominates at $a > 0.75$ ($z < 0.3$)
- **Curvature** ($\Omega_k \propto a^{-2}$): Remains negligible which confirms spatial flatness of our universe

7.3.1 Matter-Radiation Equality

We calculate matter-radiation equality at:

- Analytical: $a_{\text{eq}} = 2.95 \times 10^{-4}$, $z_{\text{eq}} = 3387$
- Numerical: $a_{\text{eq}} = 2.95 \times 10^{-4}$, $z_{\text{eq}} = 3387$

Agreement between our analytical and numerical values validates our implementation. This transition period signifies the era of the first atoms along with later structure formation through gravitational collapse.

7.4 Lookback Time vs Scale Factor

Figure 7 here shows the relationship between lookback time and scale factor. The lookback time for the CMB ($z \approx 1100$, $a \approx 9 \times 10^{-4}$) is approximately 13.7 Gyr, indicating that recombination occurred only ~ 100 million years after the Big Bang. The non-linear relationship between lookback time and scale factor indicates the changing expansion rate of our universe.

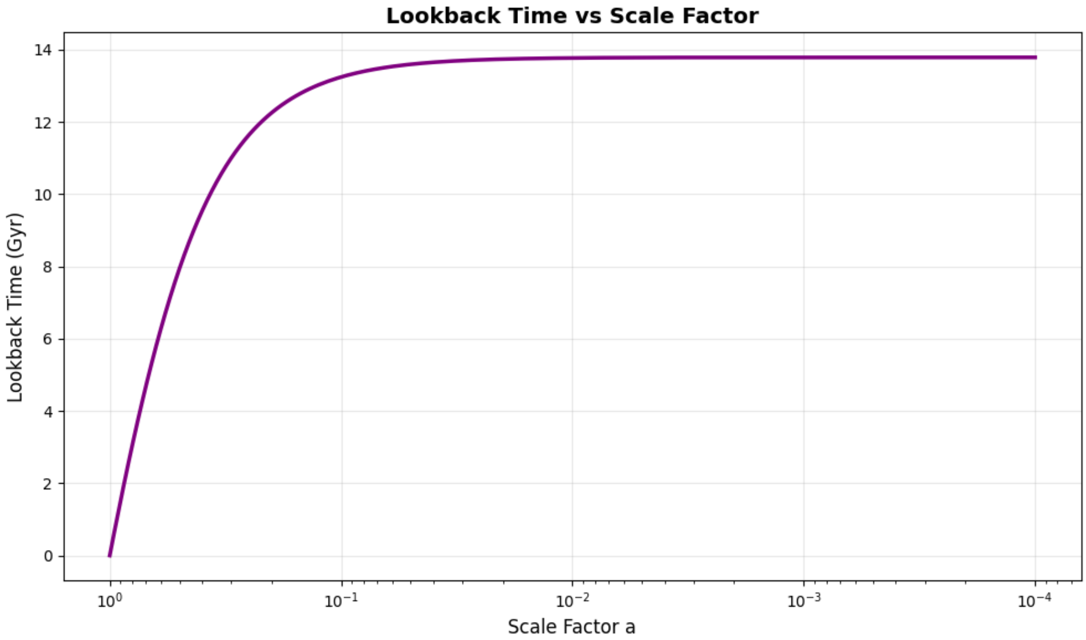

Figure 7: Lookback Time vs Scale Factor evolution

7.5 Age of the Universe

Our simulation presents us a present day age of our universe of:

$$t_0 = 13.80 \text{ Gyr} \tag{39}$$

We obtained this value through integrating the Friedmann equation from $a = 10^{-10}$ to $a = 1$. This value agrees almost identically with the Planck 2018 result of 13.787 ± 0.020 Gyr, which validates our numerical simulation.

7.6 Validation with Physical Observations

Our Numerical simulation results match with multiple observational results[6] as shown in 1:

Table 1: Comparison of simulation results with Planck 2018 observations.

Parameter	**Our Simulation**	**Planck 2018**
H_0 [km s^{-1} Mpc^{-1}]	67.66	67.4 ± 0.5
t_0 [Gyr]	13.80	13.787 ± 0.020
z_{eq}	3387	$\sim$3400
$\Omega_{m,0}$	0.311	0.315 ± 0.007
$\Omega_{\Lambda,0}$	0.689	0.685 ± 0.007

The accuracy between our numerical simulation and observed cosmological data verifies the accuracy of our simulation and the Friedmann equations.

8 Limitations and Future Outlook

Due to time constraints, we could not implement further details in our simulation. The following are planned additions currently in work which will be implemented gradually in the future:

- Simulation of the Inflation field and the inflationary epoch
- Addition of perturbations
- N-body simulation showing early gravitational collapse and structure formation
- Neutrino Masses
- Time-varying Dark Energy

9 Conclusion

Our numerical simulation has successfully managed to simulate the history and evolution of our universe. Through the implementation of numerical integration with the Friedmann equations we have managed to come up with a present day age of the universe (13.80 Gyr) and a Hubble constant (67.66 km s−1 Mpc^{-1}) that matches remarkably with observed results. Furthermore, through calculating the evolution of density parameters, we have managed to successfully identify the different epochs and transition periods of the universe with results which match theoretical predictions. Our simulation shows the incredible predictive powers of the Friedmann equations and the wider FLRW model of cosmology. This simulation also further solidifies the importance of computational physics in building model simulations which allow us to study universal processes to a remarkable degree of accuracy. Future extension of this work will focus on incorporating inflation, n-body simulation, and other important factors in order to achieve a more detailed representation of the cosmos and its evolution.

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Approval

The internship report titled “Simulating an expanding universe” submitted by Nijar Mahruz Nirjhor, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfillment of the requirements of the internship program. The internship was conducted under the supervision of **Rafid Mahbub** during the period **15 July 2025** to **15 October 2025**.

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