Generelized Global Symmetry

ICTP PWF: Physics For Bangladesh Internship

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Abstract

This report presents a structured overview of Generalized Global Symmetries, Anomalies in quantum field theory (QFT), the Strong CP problem, and one of its proposed solutions, axions. We begin by introducing higher-form symmetries as a natural generalization of ordinary (0-form) global symmetries, extending the notion of symmetry from point-like objects to those associated with extended objects such as lines, surfaces, and branes. The discussion on generalized symmetries is based on the Jena lectures on generalized global symmetries by Nabil Iqbal and the TASI lecture of Matthew Reece. After discussing the applications of these symmetries, we delve into anomalies in QFT. Primarily focusing on the Chiral (ABJ) anomaly, we then proceed with the Strong CP problem, highlighting the origin of the θ-term that appears in the QCD Lagrangian. We conclude this project by reviewing the initial part of the paper on High Quality Axion by Nathaniel Craig and Marius Kongsore.

Keywords: Anomalies, axions, branes, $\theta-$term, Peccei-Quinn mechanism, differential form.

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1 Introduction

Symmetries form the backbone of physics. In quantum field theory (QFT), the traditional notion of symmetry that spans from classical mechanics to quantum mechanics is now understood to be just one instance of a broader, more intricate structure: the *generalized global symmetries*. Recent developments have introduced the idea that extends beyond point-like transformations to include extended objects like lines or surfaces rather than particles.

Our work begins with a short review of *ordinary global symmetries*. We then move to the modern viewpoint where symmetries are interpreted as *topological operators*, which depend only on the topology of the space, not on the metric. It establishes a natural bridge toward the framework of *higher-form symmetries*.

Free electromagnetism, Goldstone's theorem, and finally the coupling of matter to electrodynamics illustrate the powerful application of generalized global symmetry. The discussion then advances to the study of *anomalies* in QFT, including the chiral anomaly and its implications for the *Strong CP problem*. The origin of the *theta term* in QCD is discussed in detail, as well as its relation to CP violation and the mechanisms that attempt to resolve the strong CP puzzle.

Finally, the focus shifts toward *extra-dimensional axion models* and their associated symmetries, inspired by recent theoretical developments. In particular, we explore how extra dimensions give rise to high-quality axions through the interplay of higher-form symmetries and their dimensional reduction, following the framework proposed by *Craig* and *Kongshore*. This provides a natural setting to understand axion's role as a *pseudo-Nambu–Goldstone* boson and as a potential solution to the strong CP problem.

2 Ordinary Global Symmetry

Ordinary global symmetry, also referred to as **0-form symmetry**, acts on local operators such as fields or particles. For a global $U(1)$ symmetry, the transformation of a field ϕ is defined as:

$$\phi(x) \to \phi'(x) = e^{i\alpha}\phi(x) \tag{2.1}$$

where α is a constant phase, independent of spacetime i.e. $e^{i\alpha} \in U(1)$ represents a global (space-time-independent) transformation.

Now, consider a charged scalar field. Its action is given by:

$$S[\phi, \phi^\dagger] = \int d^d x \left[\partial\phi^\dagger \partial\phi + V(\phi^\dagger\phi)\right] \tag{2.2}$$

This action remains invariant under the $U(1)$ symmetry. The interaction term has the general form:

$$V(\phi^\dagger\phi) = m^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2 + \dots \tag{2.3}$$

We can now promote this classical theory to a quantum one using the path integral formalism:

$$Z = \int [d\phi] \, \exp\left(-S[\phi, \phi^\dagger]\right) \tag{2.4}$$

When the field ϕ satisfies its equation of motion, we can construct a conserved Noether current:

$$J^\mu(x) = i\left(\phi^\dagger \partial^\mu \phi - \phi \partial^\mu \phi^\dagger\right) \tag{2.5}$$

This current satisfies the conservation law:

$$\partial_\mu J^\mu(x) = 0 \quad \text{[On-shell condition]} \tag{2.6}$$

However, this conservation law may not hold universally. In the presence of a charged operator, applying Noether's theorem within the path integral framework yields:

$$\langle \partial_\mu j^\mu(x)\phi(y)\ldots\rangle = i\,\delta^{(d)}(x-y)\,\langle\phi(y)\ldots\rangle \tag{2.7}$$

This expression implies that the divergence of the current vanishes everywhere except at points where a charged field operator is inserted.

Furthermore, the $U(1)$ global symmetry allows us to define a conserved charge in Lorentzian signature as:

$$Q(t) = \int_{\mathbb{R}^{d-1}} d^{d-1}x\; j^0(t,\vec{x}) \tag{2.8}$$

Performing this integral over a time-slice $\mathbb{R}^{d-1}$, the charge is measured on a $(d-1)$-dimensional spatial surface. The conservation of charge then follows from:

$$\begin{aligned}
\frac{dQ(t)}{dt} &= \int_{\mathbb{R}^{d-1}} d^{d-1}x\; \partial_0 j^0(t,\vec{x}) \\
&= -\int_{\mathbb{R}^{d-1}} d^{d-1}x\; \partial_i j^i(t,\vec{x}) \quad [\because \partial_\mu j^\mu = 0 \Rightarrow \partial_0 j^0 = -\partial_i j^i] \\
&= -j^i(t,\vec{x})\Big|_{\text{boundary}} = 0
\end{aligned} \tag{2.9}$$

This shows that charge is conserved if the current vanishes at spatial infinity. As the particle number is conserved, the world-lines can't end.

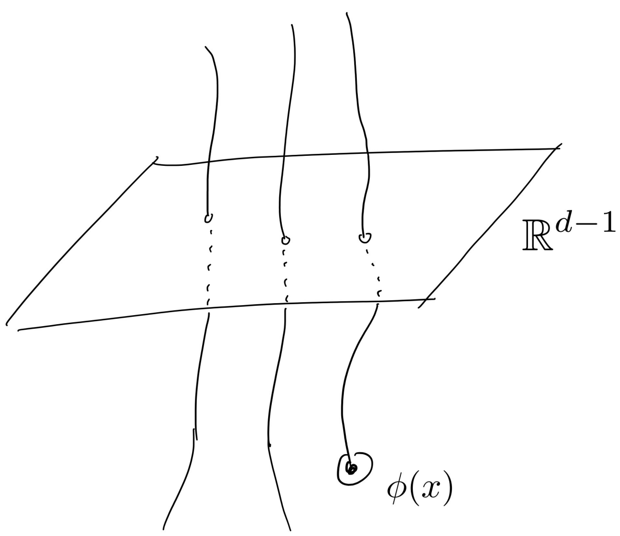

Figure 1: Particle world-lines can't end: no matter at what time-slice we evaluate Q(t) on, we get the same number.

2.1 Interacting Systems and Gauge Fields

In interacting systems, we often introduce an external gauge field a_μ coupled to the conserved current j^μ. To ensure gauge invariance under:

$$a \to a + d\Lambda \quad ; \quad \phi \to e^{i\Lambda}\phi \tag{2.10}$$

Here, Λ(x) is a function of spacetime.
we must now replace the partial derivative with a gauge-covariant derivative:

$$D_\mu\phi = \partial_\mu\phi - ia_\mu\phi \tag{2.11}$$

This modification yields the gauge-invariant action:

$$S[\phi; a] = \int d^dx \left[(D\phi)^\dagger(D\phi) + V(\phi^\dagger\phi)\right] \tag{2.12}$$

The corresponding path integral becomes:

$$Z[a] = \int [d\phi] \, \exp\left(-S[\phi, a]\right) \tag{2.13}$$

This partition function is invariant under gauge transformations of the background field a:

$$Z[a + d\Lambda] = Z[a] \tag{2.14}$$

Using this, we can compute the conserved current by taking a functional derivative:

$$j^\mu(x) = \frac{\delta Z[a]}{\delta a_\mu(x)} \tag{2.15}$$

In fact, the action of the equation (10) can be rewritten in the following form:

$$S[\phi; a] = S_0[\phi] + \int d^dx \, j^\mu(x)a_\mu(x) \tag{2.16}$$

which makes the coupling between the current and the gauge field explicit.

2.2 Applications of Global symmetry

The Old Landau Paradigm

One significant use of **global symmetries** is to classify the *phases of matter*. In the presence of symmetry, the only possible phases are: *Unbroken* and *Spontaneously broken*. The transition between the two is a *phase transition*.

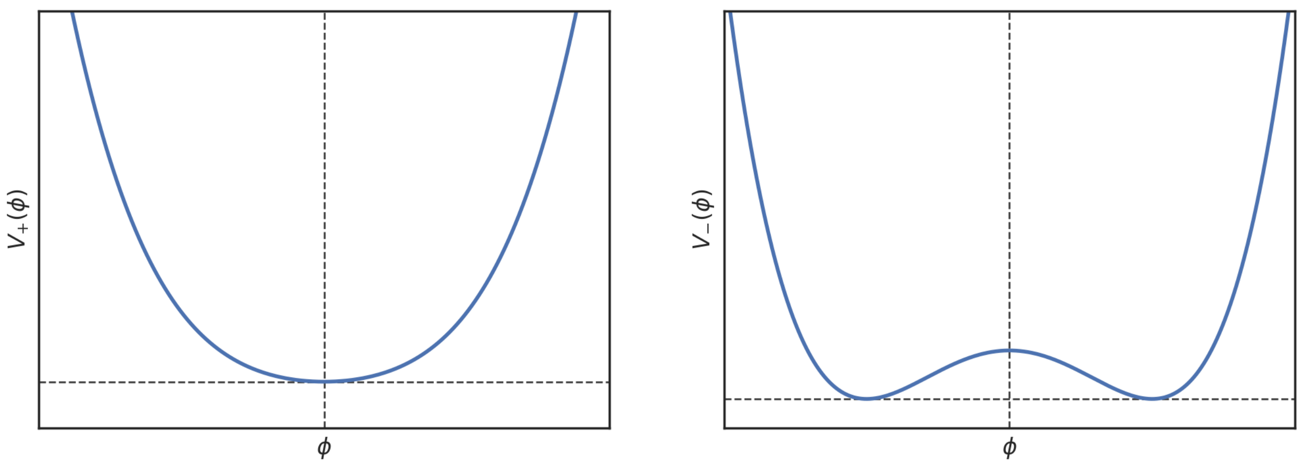

Figure 2: The Unbroken Phase (left), The Spontaneously Broken Phase (right) [1]

1. **The Unbroken Phase:** In this phase:
   - $m^2 > 0$
   - There is a *unique vacuum* in which the field ϕ is basically zero, i.e., $\langle \phi(x) \rangle = 0$
   - This vacuum is invariant under the $U(1)$ symmetry, which means the symmetry is *unbroken.*

   In this unbroken phase, the two-point function decays exponentially with distance:

   $$\langle \phi(x)\phi(0) \rangle \sim e^{-m|x|} \tag{2.17}$$

   This exponential decay arises because $\phi(0)$ creates a particle at the origin, which must survive until it is annihilated at $\phi(x)$. The cost in action due to the heavy particle explains the decay.

2. **The Spontaneously Broken Phase:** In this phase:
   - $m^2 < 0$
   - The potential has a family of minima, $\quad \phi^\dagger \phi = \frac{\lambda}{2m^2} \equiv v^2$
   - Thus the vacuum has a nonzero value, $\quad \langle \phi \rangle \neq 0 \quad ; \quad \langle \phi \rangle = v$
   - The $U(1)$ symmetry is *spontaneously broken.*

   The degrees of freedom in this broken phase can be parametrized as:

   $$\phi(x) = v(x) e^{i\theta(x)} \tag{2.18}$$

   Here v and θ describe the modulus and phase, respectively. The radial mode $v(x)$ acquires a mass due to the potential, while $\theta(x)$ is a massless **Goldstone mode** since it corresponds to the broken symmetry. The low-energy action becomes:

   $$S = \int d^d x \, v^2 (\partial\theta)^2 \cdots = \int d^d x \, v^2 (\partial^\mu \theta)(\partial_\mu \theta) + \ldots \tag{2.19}$$

   Goldstone's theorem implies non-perturbatively that if a continuous symmetry is spontaneously broken, there exists a massless mode associated with it.

   Another perspective on broken symmetry can be considered by computing:

   $$\lim_{|x|\to\infty} \langle \phi^\dagger(x)\phi(0) \rangle \sim \langle \phi^\dagger(x) \rangle \langle \phi(0) \rangle = v^2 \tag{2.20}$$

   This illustrates that in the broken phase, the two-point function saturates at large distances, indicating the presence of long-range order and broken symmetry. The field ϕ serves as the *order parameter* for the phase transition, since it behaves differently in each phase.

3. **Phase Transition:** As we can see from above, the boundary between the two phases is at $m^2 = 0$. At such a continuous transition, one can have a **conformal field theory (CFT)**, characterized by the symmetry group (e.g., $U(1)$) and the number of dimensions. The action at the critical point ($m^2 = 0$) becomes:

   $$S = \int d^d x \left(\partial\phi^\dagger \partial\phi + \frac{\lambda}{4} (\phi^\dagger \phi)^2 + \cdots \right) \tag{2.21}$$

   At criticality, the two-point function exhibits power-law behavior:

   $$\langle \phi^\dagger(x)\phi(0) \rangle \sim \frac{1}{|x|^{2\Delta}} \tag{2.22}$$

   Here Δ is called the operator dimension in CFT or critical exponent in statistical physics, and it depends on the interactions in the theory (especially for $d < 4$).

More Examples of Landau Paradigm

- U(1) Superfluid
- Magnetization
- Liquids and Solids

Physics Beyond the Landau Paradigm

Modern physics has revealed a variety of phases that go beyond the traditional Landau paradigm of symmetry-breaking framework:

- Gauge Theory
- Topological Order
- Gravity

3 Symmetries as Topological Operators

Traditionally, global symmetries in quantum field theory are defined through their action on fundamental fields and are associated with conserved currents via Noether's theorem. However, this approach relies on a specific choice of field variables and therefore fails to generalize across dual descriptions or non-Lagrangian theories.

A modern perspective redefines symmetries as codimension-1[1] topological operators[2]: often called charge defects that can be inserted in spacetime and act on other operators when crossed. In the modern framework, global symmetries are associated with codimension-1 topological operators because these are the natural objects that can separate regions of spacetime and act on local operators in a way that mimics symmetry transformations.
To make this analogous transition from fields to topological operators, we can take a step backward to a 0-form conserved current:

$$\partial_\mu j^\mu = 0 \Rightarrow d \star j = 0 \tag{3.1}$$

Similarly, on a codimension-1 manifold M_{d-1}, we can construct a charge integral of the form:

$$\mathcal{Q}(\mathcal{M}_{d-1}) = \int_{\mathcal{M}_{d-1}} d^{d-1}x\sqrt{h}\, n_\mu j^\mu = \int_{\mathcal{M}_{d-1}} \star j \tag{3.2}$$

Though this current appears to be a function of the manifold M_{d-1}, it is actually independent under small wiggles of M_{d-1}. It is called the **topological defect** of the theory, i.e., something that doesn't change under small deformations of M.

[1]This implies that it has 1 dimension less than the ambient space, e.g, a 3D volume in 4D has codimension 1.

[2]A topological operator is an object (like a line or surface operator) in a quantum field theory that does not depend on the local geometry (metric) of where it's placed. Only depends on the topology of its path or surface.

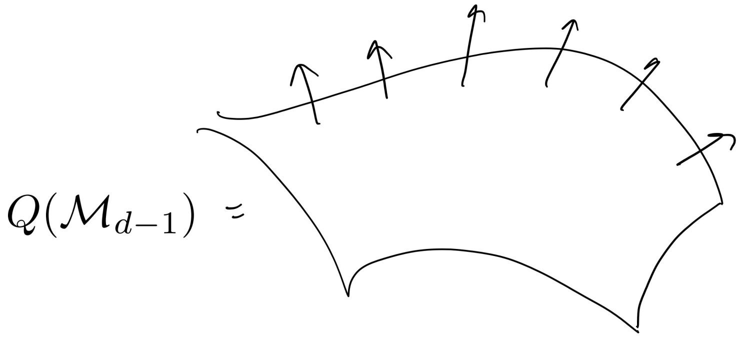

Figure 3: Integrating divergenceless current $\star j$ on a codimension-1 manifold $\mathcal{M}_{d-1}$

Hence, this notion implies that whenever a codimension-1 topological defect of the theory is obtained, the system has a normal symmetry.

3.1 Interaction of a charged defect and a charged operator

From the Ward identity, we saw that in the presence of a charge operator, the current is not conserved in normal symmetry:

$$\partial_\mu j^\mu(x)\mathcal{O}(y) = i\delta^{(d)}(x-y)\mathcal{O}(y) \tag{3.3}$$

A similar phenomenon happens for a topological defect. When we drag the topological defect M_{d-1} through the insertion of a charged local operator, it is no longer topological; it picks up a phase.

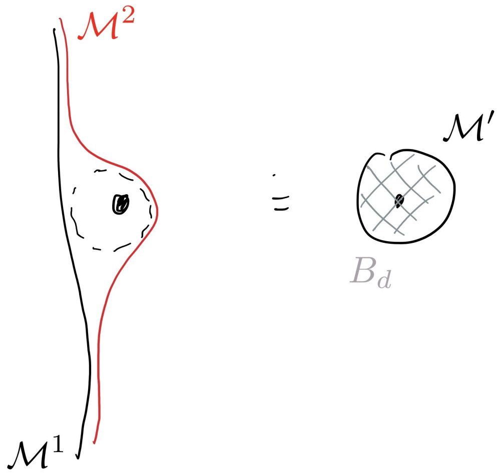

Figure 4: Deforming $\mathcal{M}_1$ to $\mathcal{M}_2$ receives a contribution only from the insertion of the local operator $\mathcal{O}(x)$.

$$\begin{aligned}(\mathcal{Q}(\mathcal{M}^1_{d-1}) - \mathcal{Q}(\mathcal{M}^2_{d-1}))\mathcal{O}(x) = \mathcal{Q}(\mathcal{M}'_{d-1})\mathcal{O}(x) = \oint_{\mathcal{M}'_{d-1}} \star j\mathcal{O}(x) \\ = \int_{B_d}(d\star j)\mathcal{O}(x) = i\int_{B_d}\delta(x-y)\mathcal{O}(x) = i\mathcal{O}(x)\end{aligned} \tag{3.4}$$

We can also define the charge operator on a $(d-1)$-dimensional manifold sphere S^{d-1} that surrounds the operator, and then shrink that sphere's dimension; it collapses into the point where the operator is located. We get:

$$\mathcal{Q}(S^{d-1})\mathcal{O}(x) = i\mathcal{O}(x) \tag{3.5}$$

Then, in the case of a finite charge rotation by an angle α, it will be:

$$U_\alpha(\mathcal{M}_{d-1}) \equiv e^{i\alpha\mathcal{Q}(\mathcal{M}_{d-1})} \quad , \quad U_\alpha(S^{d-1})\mathcal{O}(x) = e^{i\alpha}\mathcal{O}(x) \tag{3.6}$$

i.e., by collapsing the charge operator onto other local operators, a linear phase rotation of the field is generated.

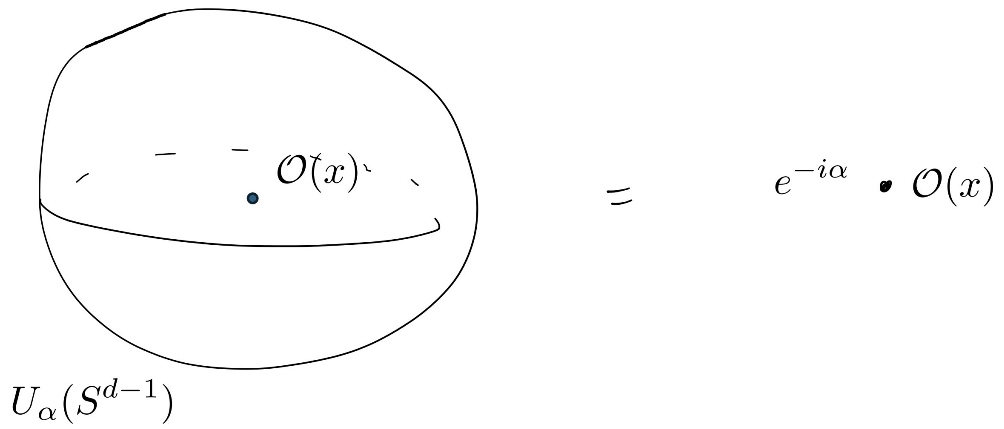

Figure 5: Collapsing the charge generator $U_\alpha(S^{d-1})$ onto $\mathcal{O}(x)$ acts as a phase rotation.

Under the group, group composition of two $U(1)$ group elements $e^{i\alpha}$ and $e^{i\beta}$, it can also be written as:

$$U_\alpha(\mathcal{M}_{d-1})U_\beta(\mathcal{M}_{d-1}) = U_{\alpha+\beta}(\mathcal{M}_{d-1}) \tag{3.7}$$

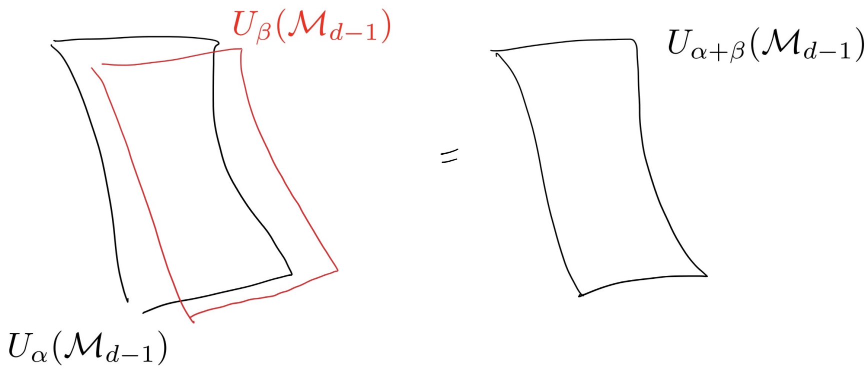

Figure 6: Fusing together two charge defects results U_α and U_β results in a combined charge defect $U_{\alpha+\beta}$.

we can also generalize this formalism to an arbitrary non-abelian global symmetry.

4 Higher Form Symmetries

Higher-form symmetries are a generalization of these normal symmetries, but instead of acting on point particles, they now act on extended objects:

- 1-form symmetry acts on line-like objects, e.g., strings or Wilson lines
- 2-form symmetry acts on surface-like objects, e.g., branes and so on.

In general, a p-form symmetry acts on p-dimensional operators and is generated by topological operators of codimension $p+1$.

4.1 1-form Symmetry

To make the transition from 0-form symmetry to p-form symmetry, we define a conserved current of the form:

$$\partial_\mu j^{\mu\nu} = 0 \quad \text{or} \quad d \star j = 0 \tag{4.1}$$

Here, $j^{\mu\nu}$ is an antisymmetric two-index current. This object is consistent with a $U(1)$ symmetry which counts strings. The extra index indicates the direction of the strings.

Similarly, a charge is defined on a codimension-2 manifold as:

$$\mathcal{Q}(\mathcal{M}_{d-2}) = \int_{\mathcal{M}_{d-2}} d^{d-2}x\sqrt{h}\, n^{\mu\nu} J_{\mu\nu} = \int_{\mathcal{M}_{d-2}} \star j \tag{4.2}$$

It is a topological defect on codimension-2.

Therefore, in 1-form global symmetry, the symmetry acts on extended objects such as line operators rather than on point-like particles. If we consider a line operator $W(C)$ defined along a one-dimensional curve $C \in \mathbb{R}^d$, then it can be interpreted as creating a string-like excitation supported along the curve C. The Ward identity associated with this symmetry takes the form:

$$\partial_\mu j^{\mu\nu}(x) W(C) = iq \int_C ds\, \frac{dx_C^\nu}{ds} \delta^{(d)}(x - X_C(s)) W(C) \tag{4.3}$$

This implies that the current is not conserved whenever it is evaluated on the worldline of the extended charged operator.

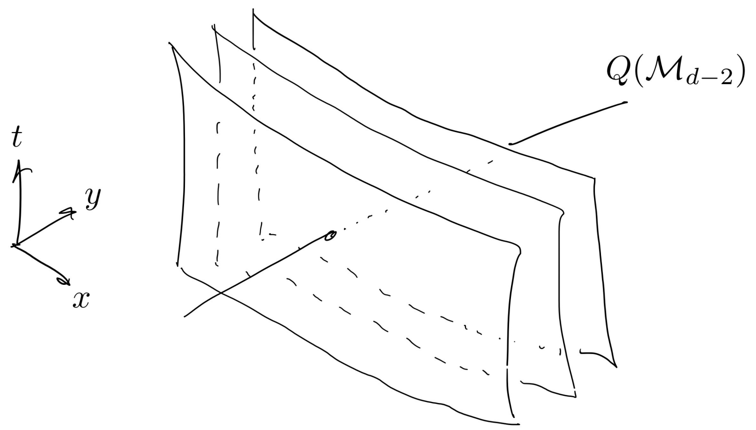

Figure 7: To count $2d$ string worldsheets, one does an integral over a codimension-2 surface, not a codimension-1 surface as for particle worldlines

We can also write this expression as an integrating form using Stokes' theorem as,

$$\int_{R^d} B_P \wedge \delta_{C_p(x)} = \int_{C_p} B_p \tag{4.4}$$

Where, $(d-p)$ form-values delta function picks up all the values whenever $x \in C_p$.
The previous expression is for the $p = 1$ case, and we can write it as

$$d \star J(x) W(C) = iq\delta_C(x) W(C) \tag{4.5}$$

Analogously, we can also construct an object that generates a field rotation on its codimension-2 manifold as:

$$U_\alpha(\mathcal{M}_{d-2}) = \exp\left(i\alpha \mathcal{Q}(\mathcal{M}_{d-2}) \right) \tag{4.6}$$

If we again drag this object through a line operator $W(C)$, it will pick up a phase:

$$U_\alpha(S^{d-2}) W(C) = e^{iq\alpha}\, W(C) \tag{4.7}$$

Physically, this indicates that the line operator acts as a charged object under the 1-form symmetry, and the current's non-conservation precisely reflects the presence of this extended charged excitation.

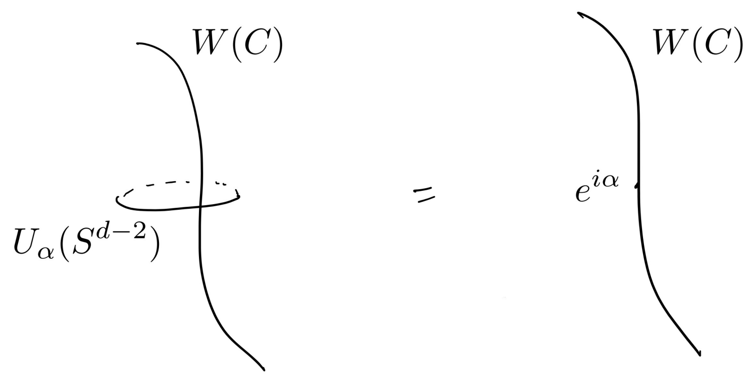

Figure 8: Collapsing the 1-form charge defect $U_\alpha(S^{d-2})$ onto a line operator acts as a phase rotation on the line.

4.2 P-form Symmetry

The notion of higher-form symmetries generalizes to any p-form symmetry:

- There exists a conserved $(p+1)$-form current $J_{\mu_1 \cdots \mu_{p+1}}$,
- Charge operators $Q(\mathcal{M}^{d-p-1})$ are defined on codimension-$(p+1)$ manifolds,
- Charged operators live on p-dimensional submanifolds C_p.

However, this generalization does not extend to non-Abelian higher-form symmetries. To argue this, we can consider the group composition of p-form symmetry charge operators:

$$U_g(\mathcal{M}_{d-p-1}) U_{g'}(\mathcal{M}_{d-p-1}) = U_{gg'}(\mathcal{M}_{d-p-1}) \tag{4.8}$$

We need to make a clear notion here that for codimension 1 operators ($p = 0$), there is a well-defined ordering (time-ordering) from the path integral, making the product well-defined.

But for higher codimension operators ($p > 0$), no natural ordering exists since the operators can be moved continuously around one another. The ambiguity in ordering implies the group elements g and g' must commute, forbidding non-Abelian structures.

However, discrete higher-form symmetries such as $\mathbb{Z}_N$ do exist and play important roles, and also similar to 0-form discrete symmetries, the charge operator $U_g(\mathcal{M}^{d-p-1})$ is defined with $g \in \mathbb{Z}_N$.

4.3 Phases of 1-form symmetry

To understand the phases of 1-form symmetries, we analyze the expectation value of the charged object, i.e., $\langle W(C)\rangle$.

1. **Unbroken Phase:** In the unbroken phase, for large closed curves C, we have the **area law**:

$$\langle W(C)\rangle \sim e^{-(T\,\mathrm{Area}[C])} \tag{4.9}$$

   where Area$[C]$ is the area of the minimal surface that fills in the curve C, i.e., the minimal surface S such that

$$\partial S = C \tag{4.10}$$

   This is the higher-form analog of exponential decay as seen in 0-form (ordinary) symmetry. The interpretation is that the objects the operator creates—i.e., string worldsheets—are massive with tension T, meaning the line operator creates a 2d worldsheet that must fill the curve.

2. **Spontaneously Broken Phase:** In the broken phase, we imagine the strings have "condensed", meaning the tension T has dropped to zero. This leads to a change in behavior: the line operator now satisfies a **perimeter law**:

$$\langle W(C)\rangle \sim e^{-(m\,L[C])} \tag{4.11}$$

   where $L[C]$ is the length of the curve (or more generally any local functional of the curve data).

   In this case, the correlation function factorizes as much as possible. The line operator can always be redefined by a c-number, which is a local functional of C. Thus, the line operator is essentially constant under redefinition by such c-numbers, and the parameter m becomes non-universal.

3. **Phase Transition:** If the transition is continuous, we expect a conformal field theory description, and the expectation value becomes:

$$\langle W(C)\rangle = e^{-(\ m'L[C]f(C)\ )} \tag{4.12}$$

   Where $f[C]$ is expected to depend on C in a nontrivial manner and can encode interesting universal data about the transition.

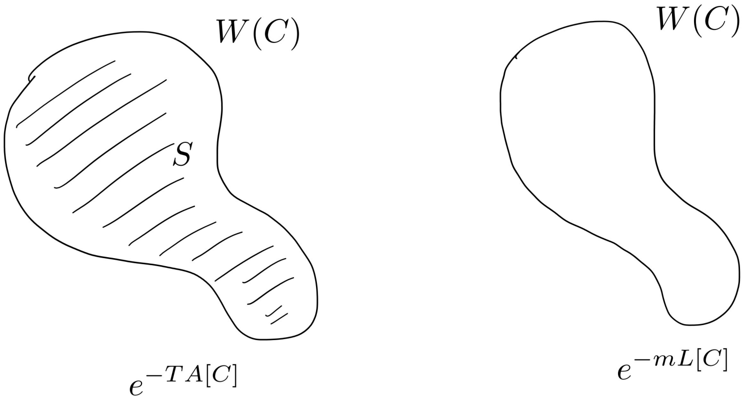

Figure 9: Area vs. perimeter laws in the spontaneous breaking of a 1-form symmetry.

Continuous transitions involving the spontaneous breaking of 1-form symmetry are rare and typically occur in a first-order manner. One known exception is $\mathbb{Z}_2$ gauge theory in 3d, which allows a second-order transition.

The structure of phases and phase transitions for 1-form symmetries parallels that of 0-form symmetries, and this analogy proves useful for classifying phases and applying similar analytical tools.

5 Applications

5.1 Free Electromagnetism

5.1.1 In D Dimension

The free Maxwell theory is the prototype of a physical theory with higher-form symmetries. The action can be defined in terms of a compact U(1) gauge field a with dimension [a]=1, i.e.

$$S[a] = \int \left(\frac{-1}{2e^2} f \wedge \star f \right) = \int d^D x \left(\frac{-1}{4e^2} f_{\mu\nu} f^{\mu\nu} \right) \tag{5.1}$$

Here, $f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu$ and the coupling constant e^2 has dimension $[e^2] = 4$-D.
The equations of motion of the action are

$$\frac{1}{e^2} d \star f = 0 \quad df = d \star (\star f) = 0 \tag{5.2}$$

These equations of motion imply that the theory has two types of higher form symmetries, namely,

1. a 1-form electric symmetries
2. a $(D-3)$-form magnetic symmetries

with currents defined by

$$J_e \equiv \frac{1}{e^2} f \quad J_m \equiv \frac{1}{2\pi} \star f \tag{5.3}$$

These symmetries will be referred to as

$$U(1)_e^{(1} \times U(1)_m^{(D-3)} \tag{5.4}$$

In general :

- $D = 2$: Only electric 1-form symmetry
- $D = 3$: Electric 1-form + magnetic 0-form
- $D = 4$: Electric 1-form + magnetic 1-form(self-dual situation)

This is discussed in detail in [2]

5.1.2 In 4 dimension

In 4d, the action of the free $U(1)$ electromagnetism is:

$$S[A] = \int d^4x \left(-\frac{1}{4e^2}F^2\right) \quad , \quad F = dA \tag{5.5}$$

The equation of motion is $\partial_\mu F^{\mu\nu} = 0$ with no source. This simple theory has two 1-form global symmetries associated with it: an electric one $U(1)_e^{(1)}$ and a magnetic one $U(1)_m^{(1)}$ with the following currents:

$$j_e^{\mu\nu} = \frac{1}{e^2}F^{\mu\nu} \quad , \quad j_m^{\mu\nu} = \frac{1}{4\pi}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{5.6}$$

In form notation:

$$J_e = \frac{1}{e^2}F \quad , \quad J_m = \frac{1}{2\pi}\star F \tag{5.7}$$

The conservation of the electric current $J_e^{\mu\nu}$ holds *on-shell*, meaning it is valid when the equations of motion are satisfied. This reflects the physical idea that electric field lines behave like continuous strings that do not end, due to the absence of electrically charged matter.

On the other hand, the magnetic current $J_m^{\mu\nu}$ is conserved as a consequence of the Bianchi identity: $\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$ Now, the charged line operator associated with $U(1)_e^{(1)}$ is the Wilson's line, i.e., the worldline of a test electric charge:

$$W_q(C) = e^{iq\int_C A} \tag{5.8}$$

In the presence of it, we obtain an expression of the current:

$$(\ d\star J_e(x)\ )W_q(C) = iq\delta_C(x)W_q(C) \tag{5.9}$$

These charges are integers q ϵ $\mathbb{Z}$.

In addition, in this setup, the action of the symmetry of the $U(1)_e^{(1)}$ 1-form on the gauge field A is simple to describe: it shifts A by a closed 1-form Λ, meaning A $\rightarrow$ A + Λ with d$\Lambda = 0$. This shift leaves the Lagrangian unchanged, preserving the symmetry. Under this transformation, a Wilson line operator acquires a phase factor:

$$W_q(C) \rightarrow e^{iq\int_C \Lambda}\ W_q(C) \tag{5.10}$$

We can also couple an external source $b_{\mu\nu}$ to this 1-form symmetry.

$$S[A; b_e] = \int d^4x \left(-\frac{1}{4e^2}(dA - b_e)^2\right) \tag{5.11}$$

The 1-form invariance is now enhanced to:

$$A \rightarrow A + \Lambda, \quad b_e \rightarrow b_e + d\Lambda \quad , \quad \Lambda \text{ arbitrary} \tag{5.12}$$

Similarly, the charged line operator under $U(1)_m^{(1)}$ is called **'t Hooft line**, $T_p(C)$: the worldline of a test magnetic monopole.

Ward identity for this current source will be:

$$(\ d\star J_m(x)\ )T_q(C) = iq\delta_C(x)\,T_q(C) \tag{5.13}$$

Now, to understand whether these symmetries are spontaneously broken or unbroken, we examine the expectation values:

Free Electrodynamics (Normal Phase)

Both expectation values follow a perimeter law:

$$\langle W_q(C)\rangle \sim \exp(-\Lambda L[C]) \tag{5.14}$$

where Λ is a UV cutoff and $L[C]$ is the length of the loop.

This implies that both 1-form symmetries, $U(1)_e^{(1)}$ and $U(1)_m^{(1)}$, are **spontaneously broken**.

This happens because the vacuum can be interpreted as a *condensate of electric and magnetic strings*. Long-wavelength excitations of these strings are identified as photons:

- The photon arises as a **Goldstone mode** of the broken 1-form symmetry.

5.2 Goldstone's Theorem

5.2.1 0-form Symmetries

Again considering the Ward identity for a charged operator $\mathcal{O}(x)$ at the origin

$$\partial_\mu j^\mu(x)\,\mathcal{O}(0) = iq\,\delta^{(d)}(x)\,\mathcal{O}(0) \tag{5.15}$$

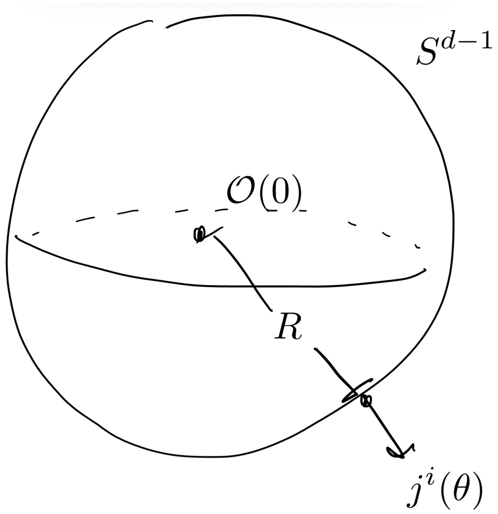

Figure 10: Geometry used for proof of 0-form Goldstone theorem.

If we integrate it using the divergence theorem we get

$$\oint_{S^{d-1}(R)} d^{d-1}\theta\, n_i j^i(\theta)\,\mathcal{O}(0) = iq\int_{B^d(R)} d^d x\,\delta^{(d)}(x)\,\mathcal{O}(0) \tag{5.16}$$

$$\tag{5.17}$$

where θ describes the angle parametritizing the S^{d-1}. Now, the right-hand side picks out the delta function at the origin, giving:

$$\oint_{S^{d-1}(R)} d^{d-1}\theta\, n_i j^i(\theta)\,\mathcal{O}(0) = iq\,\mathcal{O}(0) \tag{5.18}$$

Taking the expectation value of both sides, we obtain:

$$\left\langle \oint_{S^{d-1}(R)} d^{d-1}\theta \, n_i j^i(\theta) \, \mathcal{O}(0) \right\rangle = iq\langle \mathcal{O}(0)\rangle \tag{5.19}$$

There are two possible cases regarding the vacuum expectation value (VEV) $\langle O\rangle$ under a symmetry:

1. **Unbroken Symmetry**: If $\langle \mathcal{O}\rangle = 0$, then the correlation with the current vanishes:

   $$\langle j^i(\theta)\mathcal{O}(0)\rangle = 0 \tag{5.20}$$

   In this case, no further conclusions can be drawn.

2. **Spontaneously Broken Symmetry**: If $\langle O\rangle \neq 0$, then the correlation function becomes non-zero and independent of the radius R. By spherical symmetry, it takes the form:

   $$\langle j^i(\theta)\mathcal{O}(0)\rangle \sim \frac{iq\langle \mathcal{O}\rangle n^i}{R^{d-1}} \tag{5.21}$$

   This power-law decay in correlation is similar to the r^{-2} potential derived via Gauss's law in electromagnetism. The existence of such long-range behavior implies the presence of a gapless mode in the theory—this is the **Goldstone mode**.

5.2.2 Higher Form Symmetries

Similar proof arises for the higher form symmetry case. Ward identity for a p-form symmetry:

$$(d \star J(x))W(C) = iq\delta_C(x)W(C) \tag{5.22}$$

In spontaneously broken phase W(C) obeys a perimeter law

$$\langle W(C)\rangle \sim \exp(-m \, \mathrm{Perimeter}[C]) \tag{5.23}$$

Here, "perimeter" denotes the p-volume of the submanifold C. As this is a local functional of the geometric data characterizing C, we can define a new charged operator $\overline{W}(C)$ that strips this off,

$$\overline{W}(C) \equiv \exp(+m\mathrm{Perimeter}[C])W(C) \tag{5.24}$$

$\overline{W}(C)$ satisfies the same Ward identity as W(C), i.e.

$$d \star J(x)\overline{W}(C) = iq\delta_C(x)\overline{W}(C) \tag{5.25}$$

If we integrate over the B^{d-p} on the left hand side we have,

$$\int_{B^{d-p}} d \star J(x)\overline{W(C)} = \oint_{S^{d-p-1}(R)} \star J(\theta)\overline{W}(C) \tag{5.26}$$

where θ is a point on the S^{d-p}. On the right hand side the delta function contributes from a single point, we have

$$\oint_{S^{d-p}(R)} \star J(\theta)\overline{W}(C) = iq\overline{W}(C) \tag{5.27}$$

Finally, we take the expectation value:

$$\left\langle \int_{S^{d-p}(R)} \star J(\theta)\overline{W}(C) \right\rangle = iq\langle \overline{W}(C)\rangle \tag{5.28}$$

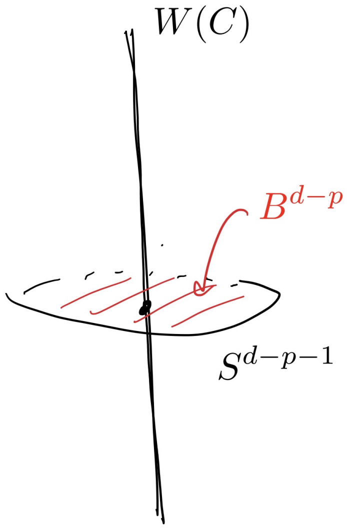

Figure 11: Geometry used for proof of p-form Goldstone theorem.

If the expectation value $\langle \overline{W}(C)\rangle$ of a surface operator is nonzero, then the correlation function between the Hodge dual current and the operator behaves as:

$$\langle \star J(\theta)\,\overline{W}(C)\rangle \sim \frac{iq}{R^{d-p-1}} \tag{5.29}$$

This power-law correlation implies the existence of a **gapless Goldstone mode**, generalizing the ordinary Goldstone theorem to **higher-form symmetries**.

The effective low-energy theory for this Goldstone mode is a p-form gauge theory:

$$S = \int d^d x\, v^2 (dB_p)^2 \tag{5.30}$$

where B_p is a dynamical p-form field. The surface operator is given by:

$$W(C) = \exp\left(iq \int_C B_p\right) \tag{5.31}$$

- For $p = 1$, this recovers **ordinary electromagnetism**, where $B_1 = A$, and $W(C)$ is the **Wilson line**.
- If the symmetry is magnetic (involving J_m), the Goldstone mode is the **magnetic photon** $\tilde{A}$, and the corresponding operator is the **'t Hooft line**, related to electric-magnetic duality.

5.3 Electrodynamics coupled to matter

If we couple the electric matter with the free photon which is the Goldstone mode of a spontaneously broken 1-form symmetry, we get an action of the form:

$$S[\phi_i, A] = \int \mathrm{d}^4 x \left(\frac{1}{4e^2}F^2 + (D\phi)^\dagger(D\phi) + V(\phi^\dagger\phi)\right) \tag{5.32}$$

With the usual gauge-covariant derivative $D_\mu\ \phi = \partial_\mu\ \phi$ - iq $A_\mu\ \phi$

$$V(\phi^\dagger\phi) = m^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2 + \dots \tag{5.33}$$

For the q=1 case, the equations of motion say that

$$\partial_\mu j_e^{\mu\nu} = iq\left(\phi^\dagger D^\nu\phi - \phi D^\nu\phi^\dagger\right) = J^\nu_{\text{el}} \tag{5.34}$$

I.e. in presence of the gauge field, the elctric 1-form symmetry $U(1)^1_e$ has been explicitly broken as the corresponding current is no longer conserved. Physically, this suggestes that electric field lines can now end on free elctric charges.
The magnetic 1-form symmetry $U(1)^{(1)}_m$, whose current is :

$$j_m^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{5.35}$$

remains, as there are no dynamical magnetic monopoles in this theory; surely it may still be spontaneously broken.

Spontaneous Breaking of EM 1-form Symmetry

1. Coulomb Phase ($m^2 > 0$)

- The scalar field ϕ is uncondensed.
- At long distances, the field ϕ is energetically suppressed; the theory effectively becomes free electromagnetism.
- The **electric 1-form symmetry** emerges in the infrared and is **spontaneously broken**.
- The **magnetic 1-form symmetry** is also **spontaneously broken**.
- Emergent long-distance symmetries are generic for 1-form symmetries, though rare for 0-form ones.

2. Higgs Phase ($m^2 < 0$)

- The scalar field condenses: $\phi(x) = v(x)e^{i\theta(x)}$, resulting in a Higgs (superconducting) phase.
- The low-energy effective action is:

$$S = \int d^4x \left[\frac{1}{4e^2}F^2 + v_0^2(d\theta - A)^2 + (dv)^2 + m_H^2(v - v_0)^2 + \cdots\right] \tag{5.36}$$

- The spectrum is **fully gapped**.
- The **electric 1-form symmetry is explicitly broken**.
- To analyze the **magnetic 1-form symmetry**, consider a probe magnetic monopole:
  - In the Coulomb phase, magnetic field lines spread freely.
  - In the Higgs phase, magnetic flux is confined into narrow tubes — **Abrikosov flux tubes** — indicating confinement.

To describe a magnetic flux tube in the Higgs phase, we solve the equations of motion derived from the action looking for a solution with nonzero magnetic flux. We work in cylindrical coordinates:

$$ds^2 = d\tau^2 + dz^2 + dr^2 + r^2 d\phi^2 \tag{5.37}$$

Scalar Field Behavior

The phase of the scalar field winds around at infinity:

$$\theta(r \to \infty) = \phi \tag{5.38}$$

To avoid singularity at the origin, the radial field must vanish near the origin and approach its vacuum value at infinity:

$$v(r \to 0) \approx 0, \quad v(r \to \infty) = v_0 \quad \theta(r \to) = \phi \tag{5.39}$$

This interpolation occurs over a length scale $\sim \frac{1}{m_H}$.

Gauge Field Behavior

Because of the kinetic term $(\partial_\phi \theta - A_\phi)^2$, the gauge field must asymptotically follow the scalar phase:

$$A_\phi(r \to \infty) = 1 \tag{5.40}$$

Magnetic Flux Quantization

The total magnetic flux stored in the flux tube is:

$$\int_{\mathbb{R}^2} F = \oint_{S^1(\infty)} d\phi\, A_\phi = 2\pi \tag{5.41}$$

This flux is quantized due to the integer winding of θ. There is no way to generate half-integer fluxes within this phase.

't Hooft Line

Now, inserting a 't Hooft line which represents a magnetic monopole, also acts as the endpoint of an *Abrikosov vortex* — a magnetic flux tube. Inserting such a monopole creates magnetic flux that is confined into a 2D worldsheet. This leads to an **area law** behavior:

$$\langle T(C) \rangle \sim \exp(-t_p \, \text{Area}[C]) \tag{5.42}$$

This implies that the magnetic 1-form symmetry is unbroken in the Higgs phase.

Although it is commonly stated that gauge symmetry is spontaneously broken in the Higgs phase, this notion is problematic beyond perturbation theory because gauge symmetry is not a true symmetry— it is a redundancy in our description. As a result, there is no observable order parameter for its breaking.

The realization of electric ($U(1)_e^{(1)}$) and magnetic ($U(1)_m^{(1)}$) **1-form symmetries** in various phases of 4D $U(1)$ gauge theory is summarized in the table below:

Phase	$U(1)_e^{(1)}$	$U(1)_m^{(1)}$
Free electrodynamics	Spontaneously broken	Spontaneously broken
Massive electrically charged matter	Explicitly broken (emergent in IR)	Spontaneously broken
Higgs phase	Explicitly broken	Unbroken
Massive magnetically charged matter	Spontaneously broken	Explicitly broken (emergent in IR)
Confined phase	Unbroken	Explicitly broken

Table 1: Realization of 1-form symmetries in different phases of 4d $U(1)$ gauge theory

In theories with minimal electric charge $q > 1 \in \mathbb{Z}$, the electric 1-form symmetry is explicitly broken to a discrete subgroup:

$$U(1)_e^{(1)} \to \mathbb{Z}_q^{(1)} \tag{5.43}$$

Electric flux tubes with less than one unit of minimal charge cannot be screened, so their number is conserved modulo q.

6 Anomalies in Quantum Field Theory

Anomalies play a fundamental role in quantum field theory: they often determine whether a theory is mathematically self-consistent and physically realizable, thereby serving as a guiding principle for selecting acceptable theories. In many cases, anomalies are tied to the appearance of new quantum numbers, the generation of physical mass scales, and the determination of the physical spectrum. Despite their name, anomalies are not pathological accidents but rather a natural and essential feature of quantum field theory.

Formally, an anomaly refers to the situation where a symmetry of the classical action is broken upon quantization. More precisely, if the classical Lagrangian is invariant under a given symmetry transformation, but this invariance is violated by quantum corrections, the resulting phenomenon is called an *anomaly*.

Anomalies are generally divided into two broad categories: *internal* and *external* anomalies.

- In the case of an **internal anomaly**, gauge invariance of the classical Lagrangian is destroyed at the quantum level. Such anomalies render the theory non-renormalizable and inconsistent. To obtain a viable theory, one requires a special assignment of fields (or representations) in the Lagrangian so that all gauge anomalies cancel. This cancellation condition is famously satisfied in the Standard Model.
- An **external anomaly**, on the other hand, refers to the violation of a symmetry associated with external currents or sources, rather than gauge invariance. These anomalies do not necessarily make the theory inconsistent but can have important physical consequences.

In quantum chromodynamics (QCD), anomalies play a particularly significant role. Two prominent examples are:

1. **Axial (chiral) anomaly** was first discussed in [3], which explains the anomalously fast decay of the neutral pion $\pi^0 \to \gamma\gamma$, and

2. **Scale anomaly**, which reflects the breaking of classical scale invariance[3] by quantum effects, leading to the emergence of the QCD scale Λ_{QCD} [4].

Both of these anomalies are closely linked to the short-distance (high-momentum) structure of QCD and to the necessity of introducing regularization. Since no regularization scheme exists that preserves all classical symmetries simultaneously, certain symmetries are inevitably broken at the quantum level. The evidence for these anomalies originally emerged from perturbative calculations.

6.1 The Chiral Anomaly

In theoretical physics, the *chiral anomaly* refers to the unexpected non-conservation of a chiral current. The phenomenon first appeared in studies of the decay of the neutral pion (π^0). Early calculations within the chiral model, based on current algebra, predicted that the decay $\pi^0 \to \gamma\gamma$ should be strongly suppressed, in contradiction with experimental results.

This apparent inconsistency was resolved in 1969 by Adler [5] and independently by Bell and Jackiw [3], who showed that quantum corrections spoil the conservation of the axial current. This effect, now known as the Adler–Bell–Jackiw (ABJ) anomaly, provides a striking example of a symmetry that is respected in the classical theory but violated in the quantum theory.

The chiral anomaly plays an important role in both quantum electrodynamics (QED) and quantum chromodynamics (QCD), where it is closely connected to the $U(1)_A$ problem and the absence of a ninth Goldstone boson [6].

6.2 The $U(1)_A$ Problem of QCD

A simple and instructive example of a quantum anomaly arises in the case of a massless Dirac fermion in $d = 3 + 1$ dimensions coupled to an electromagnetic gauge field [7]. The fermion action reads

$$S = \int d^4x \, \bar{\psi} \not{D} \psi, \tag{6.1}$$

where $\not{D}$ is the gauge-covariant derivative. If the gauge field is dynamical, one should add the Maxwell action; alternatively, one may consider A_μ as a fixed background field.

The action (6.1) possesses two global symmetries:

- A **vector rotation**, $\psi \to e^{i\alpha}\psi$, with associated current

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \tag{6.2}$$

  which couples to the gauge field through $A_\mu j^\mu$. Gauge invariance under $A_\mu \to A_\mu + \partial_\mu \alpha$ requires conservation of the vector current, $\partial_\mu j^\mu = 0$.

- An **axial rotation**, $\psi \to e^{i\alpha\gamma_5}\psi$, with corresponding current

$$j^\mu_A = \bar{\psi}\gamma^\mu\gamma_5\psi. \tag{6.3}$$

  Classically, Noether's theorem implies $\partial_\mu j^\mu_A = 0$. However, in the quantum theory this symmetry is broken.

[3] Classically, QCD has no mass scale.

Specifically, the divergence of the axial current acquires the anomalous contribution [5, 3]:

$$\partial_\mu j^\mu_A = \frac{e^2}{16\pi^2}\,\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{6.4}$$

where $F_{\mu\nu}$ is the electromagnetic field strength. This is the celebrated *chiral anomaly*, also known as the Adler–Bell–Jackiw (ABJ) anomaly.

The anomaly implies that in the presence of parallel electric and magnetic fields, the axial charge density is not conserved. This mechanism plays a central role in QCD, where the chiral $U(1)_A$ symmetry is anomalous due to the ABJ anomaly. Because this symmetry is explicitly broken at the quantum level, it does not give rise to a corresponding Goldstone boson, thereby resolving the so-called $U(1)$ problem of QCD [6, 8].

6.3 Deriving the Chiral Anomaly

There are two different approaches to compute the anomaly.

1. Path integral approach through Noether's current and Ward Identity (Fujikawa's Method[7]). Here one considers the anomaly as arising from a lack of invariance of the path integral measure. [9]

2. Perturbative approach which generates the famous Triangle Diagram [9]. Indeed, this is how the anomaly was first discovered.

6.3.1 Derivation

At first, we consider QCD with both light quarks q_l of mass m_l and a heavy quark Q of mass $M \gg \Lambda_{\text{QCD}}$. Then the generating functional is

$$Z = \int \mathcal{D}A_\mu\, \mathcal{D}q_l\, \mathcal{D}\bar{q}_l\, \mathcal{D}Q\, \mathcal{D}\bar{Q}\;\exp\Big(i\int d^4x\, \mathcal{L}_{\text{QCD}}\Big), \tag{6.5}$$

where the QCD Lagrangian reads

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} + \bar{q}_l(i\not{D} - m_l)q_l + \bar{Q}(i\not{D} - M)Q. \tag{6.6}$$

If the heavy quark mass M is very large, excitations of Q are not accessible at low energies. Therefore, we can *integrate out* the heavy quark fields from the path integral. The heavy quark action is quadratic:

$$S_Q = \int d^4x\, \bar{Q}(i\not{D} - M)Q. \tag{6.7}$$

This situation is directly analogous to the finite-dimensional Gaussian integral:

$$\int d^n x\;\exp\Big(-\tfrac{1}{2}x^T A x\Big) \;\;\propto\;\; (\det A)^{-1/2}, \tag{6.8}$$

where A is a positive-definite $n \times n$ matrix.
Hence, the Gaussian functional integral over $(Q, \bar{Q})$ gives

$$\int \mathcal{D}Q\, \mathcal{D}\bar{Q}\; e^{iS_Q} \;=\; \det\Big(i\not{D} - M\Big). \tag{6.9}$$

After integrating out Q, the generating functional becomes

$$Z = \int [dA_\mu] \int [dq]\,[d\bar{q}]\, e^{iS_{eff}} \tag{6.10}$$

Here, S_{eff} is the effective action which describes the interactions of gluons at scales well below M.
Now, we can consider a term in the effective action with two external gauge bosons which results in a famous triangle diagram showing a loop of fermions interacting with one axial gauge field and two vector gauge fields.
One node of the triangle is given by the current j_{μ_5} with momentum q and the other two nodes represent photons given by the vector current j_μ with momenta p_1 and p_2.
The current j_{μ_5} has two fermions and one photon. The fermions which run in the loop give two of the internal lines of the diagram while the external legs are currents associated to the U(1) symmetry.
Here, two triangular diagrams appear because of the possible crossings of the two external gauge bosons connected to the vector current vertex.

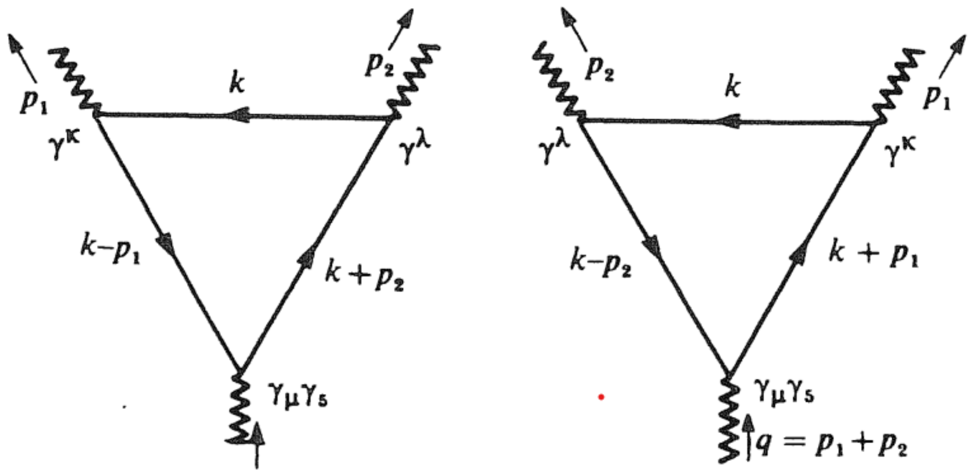

Figure 12: The triangle diagram associated with the four dimensional anomaly[10].

Therefore, the anomalous term in the action is given by

$$\delta\mathcal{L}_{\text{eff}} = -i\frac{\theta}{2}g^2M\,\text{Tr}(T^aT^b)\int\frac{d^4k}{(2\pi)^4}\text{Tr}\left[\gamma_5\frac{1}{\not{k}+\not{q}_1-M}\not{\epsilon}_1\frac{1}{\not{k}-M}\not{\epsilon}_2\frac{1}{\not{k}-\not{q}_2-M}\right] \tag{6.11}$$

$$= -i\frac{\theta}{2}g^2M\text{Tr}(T^aT^b)\int\frac{d^4k}{(2\pi)^4}\text{Tr}\left[\gamma_5\frac{\not{k}+\not{q}_1+M}{(k+q_1)^2-M^2}\not{\epsilon}_1\frac{\not{k}+M}{k^2-M^2}\not{\epsilon}_2\frac{\not{k}-\not{q}_2+M}{(k-q_2)^2-M^2}\right] \tag{6.12}$$

For small q, we can neglect the q-dependence in the denominator and then using Feynman parameters:

$$= -i\frac{\theta}{2}g^2M\text{Tr}(T^aT^b)\int\frac{d^4k}{(2\pi)^4}\text{Tr}\left[\gamma_5\frac{\not{k}+\not{q}_1+M}{k^2-M^2}\not{\epsilon}_1\frac{\not{k}+M}{k^2-M^2}\not{\epsilon}_2\frac{\not{k}-\not{q}_2+M}{k^2-M^2}\right] \tag{6.13}$$

$$= -i\frac{\theta}{2}g^2M\,\text{Tr}(T^aT^b)\int d\alpha_1 d\alpha_2\int\frac{d^4k}{2\pi}\text{Tr}\,\gamma_5\left[\frac{(\not{k}-\alpha_1\not{q}_1+\alpha_2\not{q}_2+\not{q}_1+M)\not{\epsilon}_1}{(k^2-M^2)^3}\right.$$

$$\left.\times\,(\not{k}-\alpha_1\not{q}_1+\alpha_2\not{q}_2+M)\not{\epsilon}_2(\not{k}-\alpha_1\not{q}_1+\alpha\not{q}_2-\not{q}_2+M)\right] \tag{6.14}$$

We can drop terms linear in k and only terms proportional to

$$\mathrm{Tr}\Big[\gamma_5 \not{q}_1 \not{q}_2 \not{\epsilon}_1 \not{\epsilon}_2\Big] = -i\epsilon_{\mu\nu\rho\sigma} q_1^\mu q_2^\nu \epsilon_1^\rho \epsilon_2^\sigma \tag{6.15}$$

will survive. After performing the integrals over the α's the effective interaction becomes:

$$\delta\mathcal{L}_{\text{eff}} = g^2 M^2 \theta \mathrm{Tr}(T^a T^b) \epsilon_{\mu\nu\rho\sigma} q_1^\mu q_2^\nu \epsilon_1^\rho \epsilon_2^\sigma \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - M^2)^3}. \tag{6.16}$$

Evaluation of the Standard Loop Integral(Using the dimensional regularization formula)

We wish to evaluate the integral:

$$I = \int \frac{d^4k}{(2\pi)^4} \, \frac{1}{(k^2 - M^2)^3}. \tag{6.17}$$

To evaluate such integrals, we perform a **Wick rotation**:

$$k^0 \;\rightarrow\; ik^4, \tag{6.18}$$

which transforms the Minkowski metric to Euclidean space:

$$k^2 \;\rightarrow\; -k_E^2. \tag{6.19}$$

The measure transforms as:

$$d^4k \;\rightarrow\; i\, d^4k_E, \tag{6.20}$$

and the integral becomes:

$$I = i \int \frac{d^4k_E}{(2\pi)^4} \, \frac{1}{(k_E^2 + M^2)^3}. \tag{6.21}$$

Now, the general scalar integral in d-dimensions is:

$$\int \frac{d^dk}{(2\pi)^d} \, \frac{1}{(k^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \, \frac{\Gamma(n - d/2)}{\Gamma(n)} \, \Delta^{(d/2)-n}, \qquad \Delta > 0. \tag{6.22}$$

For our case: $d = 4$; $n = 3$; $\Delta = M^2$
Upon substituting:

$$I = i \, \frac{1}{(4\pi)^2} \, \frac{\Gamma(3-2)}{\Gamma(3)} \, (M^2)^{2-3}, \tag{6.23}$$

$$= i \, \frac{1}{(4\pi)^2} \, \frac{\Gamma(1)}{\Gamma(3)} \, (M^2)^{-1}. \tag{6.24}$$

We know:

$$\Gamma(1) = 1, \qquad \Gamma(3) = 2! = 2. \tag{6.25}$$

Thus:

$$I = i \, \frac{1}{(4\pi)^2} \, \frac{1}{2} \, \frac{1}{M^2}, \tag{6.26}$$

$$= \frac{i}{2(4\pi)^2 M^2}. \tag{6.27}$$

Starting with

$$\delta\mathcal{L}_{\text{eff}} = \frac{g^2}{M^2}\,\theta\,\mathrm{Tr}(T^aT^b)\,\epsilon_{\mu\nu\rho\sigma}\,q_1^\mu q_2^\nu\,\varepsilon_1^\rho\varepsilon_2^\sigma \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-M^2)^3}, \tag{6.28}$$

$$\delta\mathcal{L}_{\text{eff}} = \frac{g^2M^2\theta}{32\pi^2M^2}\mathrm{Tr}(T^aT^b)\epsilon_{\mu\nu\rho\sigma}q_1^\mu q_2^\nu\varepsilon_1^\rho\varepsilon_2^\sigma. \tag{6.29}$$

The momentum and polarization structure reconstructs the field-strength tensors:

$$\epsilon_{\mu\nu\rho\sigma}q_1^\mu\varepsilon_1^\rho q_2^\nu\varepsilon_2^\sigma = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{a\,\mu\nu}F^{b\,\rho\sigma}. \tag{6.30}$$

Including the combinatorial factor of two from exchanging the gauge bosons and using $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$ cancels extra factors, yielding the gauge-invariant form:

$$\boxed{\delta\mathcal{L}_{\text{eff}} = \frac{\theta}{32\pi^2}\mathrm{Tr}\big(F_{\mu\nu}\tilde{F}^{\mu\nu}\big)} \quad ; \quad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{6.31}$$

6.4 Explanation of the Large Mass of the η' Meson

The unexpectedly large mass of the η' meson is a direct consequence of the chiral (axial) anomaly in Quantum Chromodynamics (QCD) [11, 12].

Classically, for three light quark flavors (u, d, s), QCD possesses an approximate $U(3)_L \times U(3)_R$ chiral symmetry. When this symmetry is spontaneously broken, it should produce nine massless Goldstone bosons — eight corresponding to the pseudoscalar octet (π, K, η) and one flavor-singlet pseudoscalar, which would be associated with the η' meson.

If this symmetry were exact, the η' should have a mass comparable to that of the pion ($\sim$ 140 MeV) or the kaon ($\sim$ 495 MeV). However, quantum effects break the axial $U(1)_A$ part of the symmetry due to the axial anomaly,

$$\partial_\mu J_5^\mu = \frac{g_s^2}{16\pi^2}G_{\mu\nu}\tilde{G}^{\mu\nu}, \tag{6.32}$$

where $G_{\mu\nu}$ is the gluon field strength and $\tilde{G}_{\mu\nu}$ its dual.

This anomaly couples the singlet pseudoscalar state to the topological structure of the QCD vacuum (such as instantons), generating an additional large mass component for the η'. As a result, the η' acquires a mass of about 958 MeV, much heavier than the other pseudoscalar mesons, which remain relatively light because they are protected by the approximate $SU(3)_A$ symmetry.

Thus, the large mass of the η' meson is a clear manifestation of the deep interplay between quantum anomalies and the non-trivial topology of the QCD vacuum.

7 The Strong CP Problem

The Strong CP problem is one of the most intriguing puzzles in modern particle physics, lying at the intersection of quantum chromodynamics (QCD), symmetries, and beyond Standard Model physics. While the Standard Model successfully incorporates the violation of charge conjugation and parity (CP) symmetry in the weak interactions, the strong interactions, governed by QCD, appear to conserve CP with remarkable precision. This is surprising because the QCD Lagrangian naturally allows for a CP-violating term proportional to the so-called θ angle, which arises from the non-trivial structure of the QCD vacuum and **instanton** effects. Therefore, to understand the strong CP problem, two essential concepts must first be introduced:

1. **Quantum Chromodynamics (QCD):** QCD is the part of the Standard Model of particle physics that describes how quarks interact via the strong nuclear force, mediated by gluons. QCD is responsible for holding protons and neutrons together in the atomic nucleus.
2. **CP Symmetry:** CP symmetry stands for *Charge-Parity* symmetry, where:
   - **C** (charge conjugation) refers to transforming a particle into its antiparticle.
   - **P** (parity) refers to flipping spatial coordinates, like observing the universe in a mirror.

In principle, the laws of physics should behave similarly under the combined CP symmetry. However, experiments show that CP violation does occur in the weak nuclear force, which helps explain the matter–antimatter asymmetry in the universe [13].

Surprisingly, QCD also predicts that CP violation could occur in the strong nuclear force. Specifically, one can add a CP-violating term to the QCD Lagrangian without breaking gauge invariance or renormalizability [4]:

$$\mathcal{L}_\theta = \theta \, \frac{g_s^2}{32\pi^2} \, G^a_{\mu\nu} \tilde{G}^{a\mu\nu} \tag{7.1}$$

where θ is a dimensionless parameter, g_s is the strong coupling constant, and $G^a_{\mu\nu}$ is the gluon field strength tensor with its dual $\tilde{G}^{a\mu\nu}$.

However, precise measurements of the neutron's electric dipole moment indicate that any strong CP violation is extremely small, consistent with zero. This constraint requires $\theta \lesssim 10^{-10}$, an unnaturally small value considering that theory allows θ to be of order one. This unexplained fine-tuning of the θ parameter is known as the **strong CP problem** [14].

7.1 The Origin of the Theta Term

7.1.1 Chiral transformations

For a Dirac spinor $\psi(x)$, the left- and right-handed components are defined via the chiral projectors:

$$P_L = \frac{1-\gamma^5}{2}, \quad P_R = \frac{1+\gamma^5}{2}, \quad \psi_L = P_L\psi, \quad \psi_R = P_R\psi. \tag{7.2}$$

A global (Abelian) chiral transformation is

$$\psi \longrightarrow \psi' = e^{i\alpha\gamma^5}\psi, \tag{7.3}$$

$$\bar{\psi} \longrightarrow \bar{\psi}' = \bar{\psi}e^{i\alpha\gamma^5}, \tag{7.4}$$

where $\bar{\psi} \equiv \psi^\dagger\gamma^0$. The transformation of $\bar{\psi}$ follows from

$$\bar{\psi}' = \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0 = \bar{\psi}e^{i\alpha\gamma^5}. \tag{7.5}$$

Therefore, for an infinitesimal axial transformation:

$$\delta\psi = i\alpha\gamma^5\psi, \quad \delta\bar{\psi} = i\alpha\bar{\psi}\gamma^5, \tag{7.6}$$

Now, the Dirac Lagrangian is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = \bar{\psi}i\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi. \tag{7.7}$$

Under a finite chiral rotation the kinetic term transforms as :

$$\bar{\psi}i\gamma^\mu\partial_\mu\psi \longrightarrow \bar{\psi}e^{i\alpha\gamma^5}i\gamma^\mu\partial_\mu(e^{i\alpha\gamma^5}\psi). \tag{7.8}$$

For constant α, there is no derivative acting on $e^{i\alpha\gamma^5}$.
Using $\{\gamma^5, \gamma^\mu\} = 0$:

$$e^{i\alpha\gamma^5}\gamma^\mu = \gamma^\mu e^{-i\alpha\gamma^5}. \tag{7.9}$$

Hence,

$$\bar{\psi}e^{i\alpha\gamma^5}i\gamma^\mu e^{i\alpha\gamma^5}\partial_\mu\psi = \bar{\psi}i\gamma^\mu\partial_\mu\psi. \tag{7.10}$$

The kinetic term is invariant under global chiral rotations.

Now, the mass term transforms as:

$$\begin{aligned} m\bar{\psi}\psi &\longrightarrow m\bar{\psi}e^{i\alpha\gamma^5}e^{i\alpha\gamma^5}\psi = m\bar{\psi}e^{2i\alpha\gamma^5}\psi & (7.11)\\ &= m\cos(2\alpha)\bar{\psi}\psi + im\sin(2\alpha)\bar{\psi}\gamma^5\psi, & (7.12)\end{aligned}$$

which explicitly breaks the axial symmetry unless α is an integer multiple of π. For an infinitesimal transformation $\alpha \ll 1$:

$$\cos 2\alpha \simeq 1 \quad ; \quad \sin 2\alpha \simeq 2\alpha \tag{7.13}$$

Therefore, from the align (7.12) the variation of the mass term will be

$$\delta(m\bar{\psi}\psi) = 2i\alpha m\bar{\psi}\gamma^5\psi. \tag{7.14}$$

The mass term is not invariant under chiral transformations.

Now, from the vector and psedo-vector matrices we can form two currents out of Dirac field bilinears. We define the axial current,

$$j_5^\mu = \bar{\psi}\,\gamma^\mu\gamma^5\,\psi, \qquad \{\gamma^5, \gamma^\mu\} = 0. \tag{7.15}$$

Its divergence is

$$\partial_\mu j_5^\mu = (\partial_\mu\bar{\psi})\,\gamma^\mu\gamma^5\,\psi + \bar{\psi}\,\gamma^\mu\gamma^5\,\partial_\mu\psi. \tag{7.16}$$

Using the Dirac aligns

$$\begin{aligned} (i\partial\!\!\!/ - m)\psi = 0 \quad &\Rightarrow \quad \gamma^\mu\partial_\mu\psi = -im\psi, & (7.17)\\ \bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0 \quad &\Rightarrow \quad (\partial_\mu\bar{\psi})\gamma^\mu = +im\bar{\psi} & (7.18)\end{aligned}$$

we substitute into the divergence:

$$\begin{aligned} (\partial_\mu\bar{\psi})\,\gamma^\mu\gamma^5\,\psi &= (im\bar{\psi})\,\gamma^5\,\psi = im\bar{\psi}\gamma^5\psi, & (7.19)\\ \bar{\psi}\,\gamma^\mu\gamma^5\,(\partial_\mu\psi) &= -\bar{\psi}\,\gamma^5\gamma^\mu\partial_\mu\psi = -\bar{\psi}\,\gamma^5(-im\psi) = im\bar{\psi}\gamma^5\psi. & (7.20)\end{aligned}$$

Therefore,

$$\partial_\mu j_5^\mu = im\bar{\psi}\gamma^5\psi + im\bar{\psi}\gamma^5\psi = 2im\bar{\psi}\gamma^5\psi. \tag{7.21}$$

For $m = 0$, $\partial_\mu j_5^\mu = 0$ and the axial current is classically conserved.
Also, the vector (fermion number) current $j^\mu = \bar{\psi}\gamma^\mu\psi$ remains conserved for any m.

$$\partial_\mu j^\mu = 0 \tag{7.22}$$

7.1.2 The QCD Vacuum

In most quantum field theories, the vacuum is often imagined as a single, unique "empty" ground state. However, in Quantum Chromodynamics (QCD), the theory of the strong interaction, the vacuum is far richer and more complex [15]

Unlike in Quantum Electrodynamics (QED), the vacuum state in QCD exhibits nontrivial structure. The QCD vacuum contains nonperturbative fluctuations of gluon and quark fields, which are responsible for:

- The spontaneous breaking of chiral symmetry, and
- The emergence of topological quantum numbers.

These features result in an infinitely degenerate vacuum structure. Because QCD is based on a non-Abelian gauge group, $SU(3)_C$, its gauge fields admit special configurations that cannot be smoothly deformed into one another. Such configurations are distinguished by an integer known as the **topological winding number** (or Chern–Simons number). Each value of this integer corresponds to a distinct but energetically equivalent classical vacuum state, often denoted by $|n\rangle$ where $n \in \mathbb{Z}$ labels the topological sector.[4]

7.1.3 Chiral Symmetry Breaking and the QCD Vacuum

In Quantum Chromodynamics (QCD), the Lagrangian for N_f massless quark flavors possesses the global chiral symmetry [16]

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A. \tag{7.23}$$

where $SU(2)_L$ acts on (u_L, d_L) and $SU(2)_R$ on (u_R, d_R) building together the chiral symmetry. $U(1)_V$ is associated with baryon number, whereas $U(1)_A$ is anomalous.
The axial $U(1)_A$ is broken explicitly by the Adler–Bell–Jackiw anomaly [5, 3], while the $SU(N_f)_L \times SU(N_f)_R \times U(1)_V$ symmetry remains intact at the classical level.

The true QCD vacuum, however, is far from trivial. Due to strong non-perturbative effects, it is characterized by a non-vanishing quark condensate,

$$\langle 0|\bar{\psi}\psi|0\rangle \;\neq\; 0, \tag{7.24}$$

which serves as an order parameter for chiral symmetry breaking [17, 18]. Under a chiral transformation, this condensate does not remain invariant, which indicates that the ground state fails to respect the full symmetry of the Lagrangian. Thus, the symmetry is spontaneously broken as

$$SU(N_f)_L \times SU(N_f)_R \;\longrightarrow\; SU(N_f)_V, \tag{7.25}$$

where the unbroken subgroup $SU(N_f)_V$ corresponds to simultaneous and identical transformations of left- and right-handed fields (the so-called "vector symmetry").

According to Goldstone's theorem, excitations along this manifold correspond to massless Goldstone bosons. In the QCD context, these are identified with the pseudoscalar mesons (pions, kaons, and the η). Since the up and down quark masses are small but not zero, these Goldstone bosons acquire a small mass, becoming pseudo-Goldstone bosons. Their masses are related to the quark condensate via the Gell-Mann–Oakes–Renner relation [19]:

$$m_\pi^2 f_\pi^2 \;=\; -(m_u + m_d)\,\langle 0|\bar{\psi}\psi|0\rangle. \tag{7.26}$$

where,

[4] The set of all classical gauge field configurations that share the same winding number n.

- m_π is the pion mass,
- $f_\pi \simeq 93$ MeV is the pion decay constant,
- m_u, m_d are the up- and down-quark masses,
- $\langle 0|\bar{\psi}\psi|0\rangle$ is the quark condensate in the QCD vacuum.

Physical meaning:

1. In the chiral limit $m_u, m_d \to 0$, QCD has an approximate $SU(2)_L \times SU(2)_R$ chiral symmetry. This symmetry is *spontaneously broken*[5] by the nonzero condensate $\langle\bar{\psi}\psi\rangle \neq 0$, producing massless Goldstone bosons, which are the pions.
2. For real QCD, the quark masses are small but nonzero. This *explicit breaking*[6] of chiral symmetry gives the pions a small but finite mass.

Symmetry	**Status after QCD vacuum forms**	**Physical Meaning**
$SU(3)_L \times SU(3)_R$	Broken $\to SU(3)_V$	8 Goldstone bosons (π, K, η_8)
$U(1)_V$	Unbroken	Conserved baryon number
$U(1)_A$	Broken by anomaly	η' is massive, no Goldstone

Table 2: Chiral symmetry breaking in QCD and its consequences.

In this way, the nontrivial structure of the QCD vacuum, through the quark condensate, is directly responsible for spontaneous chiral symmetry breaking and the existence of a degenerate set of vacua, from which the low-energy phenomenology of hadrons emerges.

7.1.4 Arrival of Instantons

However, quantum mechanics allows tunneling between these vacua. Such tunneling events are described by instantons, non-perturbative field configurations in Euclidean spacetime. Those tunnelings result the so called θ vacuum. Since tunneling is possible, the true vacuum of QCD is not just one of these states, but a coherent superposition of all of them.

The realization of *instantons* in non-Abelian gauge theories revolutionized the understanding of the QCD vacuum structure. This was first introduced in the context of Yang–Mills theory by Belavin, Polyakov, Schwartz, and Tyupkin [20].

Mathematically, instantons are self-dual (or anti-self-dual) field configurations, minimizing the Euclidean action [21]:

$$S_{\text{inst}} = \frac{8\pi^2}{g^2}, \tag{7.27}$$

and characterized by a topological charge:

$$Q = \frac{1}{32\pi^2} \int d^4x\, \varepsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma}, \tag{7.28}$$

[5] The symmetry is preserved in the Lagrangian but broken by the choice of vacuum.
[6] The symmetry is broken directly by terms in the Lagrangian.

which corresponds to the second Chern number of the gauge bundle over compactified Euclidean spacetime S^4 [22]

This non-trivial vacuum topology naturally leads to the concept of the θ-vacuum, expressed as a coherent superposition of topological sectors:

$$|\theta\rangle = \sum_{N=-\infty}^{\infty} e^{i\theta N}|N\rangle, \tag{7.29}$$

where θ is a real parameter ($0 \leq \theta < 2\pi$).
Now, the classical QCD Lagrangian for quarks ψ and gluons A^a_μ is

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - M)\psi, \tag{7.30}$$

where $G^a_{\mu\nu}$ is the gluon field strength tensor. In the functional integral formulation, the transition amplitude between θ-vacua is [23]

$$\langle\theta|e^{-iHT}|\theta\rangle = \sum_N e^{i\theta N} \int [DA]\, e^{iS_{\text{YM}}[A]}\, \delta(Q[A] - N)\,, \tag{7.31}$$

where S_{YM} is the Yang–Mills action. The θ-dependence appears as an extra phase factor $e^{i\theta Q[A]}$ in the path integral:

$$Z_\theta = \int [DA][D\psi][D\bar{\psi}]\, \exp\left\{i \int d^4x \left(\mathcal{L}_{\text{QCD}} + \theta\, \frac{1}{32\pi^2} G^a_{\mu\nu}\tilde{G}^{a\mu\nu}\right)\right\}. \tag{7.32}$$

Thus, the presence of nontrivial topological sectors modifies the QCD Lagrangian to

$$\mathcal{L}_{\text{QCD}} \quad\longrightarrow\quad \mathcal{L}_{\text{QCD}} + \mathcal{L}_\theta, \tag{7.33}$$

where

$$\mathcal{L}_\theta = \theta\, \frac{1}{32\pi^2} G^a_{\mu\nu}\tilde{G}^{a\mu\nu}. \tag{7.34}$$

This θ-term is a total derivative in perturbation theory, but becomes physically relevant due to instanton effects. Importantly, it violates P and CP symmetries, and its smallness in nature gives rise to the *strong CP problem*.

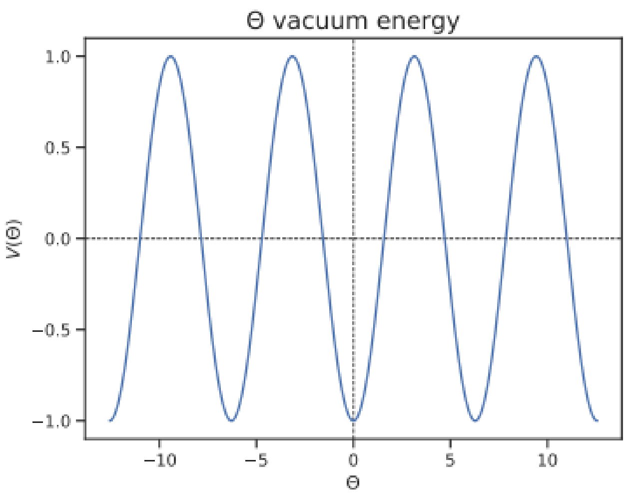

Figure 13: The vacuum in QCD has a degeneracy in the sense that it depends on θ as a periodic function on θ. Each of the minima of this function is an allowed vacuum.These are known as *theta vacua* or *θ-vacua* [16].

7.2 Properties of the Theta

Three important properties of this term [16]:

- **Total derivative:** This term is a total derivative since $\epsilon_{\mu\nu\rho\sigma}G^A_{\mu\nu}G^A_{\rho\sigma} = \partial^\mu K_\mu$ with K_μ The Chern-Simons current is defined as:

$$k^\mu = \epsilon^{\mu\nu\rho\sigma}\left(G^A_{\mu\nu}G^A_{\rho\sigma} - \frac{g_s}{3}f^{ABC}G^A_\nu G^B_\rho G^C_\sigma\right) \tag{7.35}$$

  This means that this term is not relevant in perturbation theory.

- **CP violation:** This term violates CP. The easiest way to see it is that this term has the same structure as the QED case (abelian) for which it is just E · B which changes sign under time reversal (and parity) since E being a polar vector does not change sign under time reversal, but B does. By the CPT theorem, if it violates T, it violates CP. We can expand the sum over all permutations of the indices. Using the antisymmetry of both $F^{\mu\nu}$ (i.e., $F^{\mu\nu} = -F^{\nu\mu}$) and $\epsilon^{\mu\nu\rho\sigma}$, we find that the sum[24] simplifies to four identical terms:

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} &= \sum_{i,j,k}\left(\epsilon^{0ijk}F_{0i}F_{jk} + \epsilon^{i0jk}F_{i0}F_{jk} + \epsilon^{ij0k}F_{ij}F_{0k} + \epsilon^{ijk0}F_{ij}F_{k0}\right)\\
&= \sum_{i,j,k}\left(\epsilon^{0ijk}F_{0i}F_{jk} + \epsilon^{0ijk}F_{0i}F_{jk} + \epsilon^{0ijk}F_{0i}F_{jk} + \epsilon^{0ijk}F_{0i}F_{jk}\right)\\
&= 4\sum_{i,j,k}\epsilon^{0ijk}F_{0i}F_{jk}.
\end{aligned}$$

  Now, we substitute the field components and use the relation $\epsilon^{0ijk} = \epsilon^{ijk}$ for the 3D Levi-Civita symbol:

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} &= 4\sum_{i,j,k}\epsilon^{ijk}F_{0i}F_{jk}\\
&= 4\sum_{i,j,k,l}\epsilon^{ijk}(-E_i)(-\epsilon_{jkl}B_l)\\
&= 4\sum_{i,l}\left(\sum_{j,k}\epsilon^{ijk}\epsilon_{jkl}\right)E_iB_l.
\end{aligned}$$

  We use the identity for the contraction of two 3D Levi-Civita symbols,

$$\sum_{j,k}\epsilon^{ijk}\epsilon_{jkl} = 2\delta^i_l, \tag{7.36}$$

  where δ^i_l is the Kronecker delta. Therefore,

$$\begin{aligned}
\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} &= 4\sum_{i,l}(2\delta^i_l)E_iB_l\\
&= 8\sum_{i,l}\delta^i_lE_iB_l\\
&= 8\sum_i E_iB_i\\
&= 8\,\mathbf{E}\cdot\mathbf{B}.
\end{aligned}$$

- **Quantization:** In fact, it needs to be taken into account to ensure a proper quantum description. A proper canonical quantization of Yang-Mills theory in terms of a complete set of gauge invariant states necessitates the presence of the θ-term. It is therefore important to keep in mind that it is not some mathematical obscurity, but quintessential for the quantization process itself.

7.3 Proposed Solutions

Several theoretical models have been proposed to explain the smallness of the θ parameter in QCD. Below, we summarize some of the most prominent ideas.

7.3.1 Massless Quark Hypothesis

One of the earliest solutions suggested was the **massless up-quark hypothesis**. In this scenario, a chiral transformation could eliminate the CP-violating θ term from the QCD Lagrangian. However, this approach is incompatible with experimental evidence showing that all quarks are massive, and lattice QCD calculations confirm that the up quark mass is nonzero.

7.3.2 Peccei–Quinn Mechanism

The most promising theoretical solution is the **Peccei–Quinn (PQ) mechanism**, proposed in 1977 by Peccei and Quinn [25]. This mechanism introduces a new global $U(1)_{\mathrm{PQ}}$ symmetry that dynamically drives the θ parameter to zero, effectively eliminating strong CP violation.

Key features of the PQ mechanism include:

- The θ parameter becomes a dynamical field rather than a fixed constant.
- This field naturally evolves to a value that cancels CP-violating effects in QCD.
- CP symmetry is effectively restored in the strong interaction.

Axion

A key prediction of the Peccei–Quinn mechanism is the existence of a new particle, the **axion**. The axion is a very light, neutral pseudoscalar boson with extremely weak interactions with ordinary matter, making it challenging to detect.

Axions have gained significant interest because:

- They solve the strong CP problem by dynamically canceling the effective θ term.
- They are a compelling candidate for cold dark matter in cosmology.

The axion field, $\theta(x)$, is a periodic dynamical field, $\theta \cong \theta + 2\pi$ couples to the gluon field strength in the QCD lagrangian [21]:

$$\mathcal{L} = \int d^4x \sqrt{|g|} \frac{1}{2} f^2 \partial_\mu \theta \partial^\mu \theta + \frac{N}{8\pi^2} \int \theta(x) tr(G \wedge G) \tag{7.37}$$

where f_a is the axion decay constant. This coupling ensures that the axion field dynamically relaxes the effective θ parameter to zero, eliminating strong CP violation.

7.3.3 Nelson–Barr Mechanism

Another interesting approach is the [26], in which CP is a fundamental symmetry of the Lagrangian. In this framework, the θ parameter is zero at tree level and is only generated radiatively, with contributions naturally small enough to evade current experimental limits. The following flowchart illustrates the implications of the theta angle for hadronic physics and BSM physics:

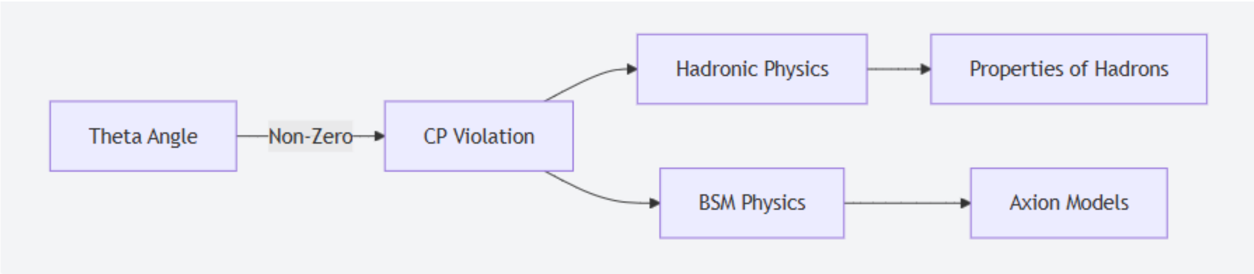

Figure 14: Implications of the theta angle.

8 Extra-dimensional Axions and Their Symmetries

- **High Quality Axion:** High-quality axions refer to axion-like particles whose global shift symmetries are exceptionally well preserved, such that any explicit symmetry-breaking effects are extremely suppressed. This high quality is crucial for the axion to effectively solve the strong CP problem without residual CP violation.

  The defining feature of an axion's quality is how well its continuous shift symmetry $\theta \to \theta$ +constant remains intact. Any small explicit breaking introduces potential terms that can spoil its ability to cancel the CP-violating term in QCD, reintroducing the strong CP problem.

- **Axion Quality Problem:** [23]The axion quality problem relates to the fact that while there is a very well-defined anomalous symmetry associated with the axion coupling, there is no symmetry associated with it. This lack of proper symmetry properties generates two issues that together are called the axion quality problem.

  1. EFTs are built by specifying the particle content and symmetries of the problem, and then writing down every coupling allowed by symmetry. Because the axion has no symmetry properties, there is no way to form the axion coupling without also including a host of other couplings.
  2. Quantum gravity is believed to break all symmetries that aren't gauged. Thus, even if one imposes the anomalous symmetry, gravitational effects will break it and the axion will obtain a separate mass term that is not centered around a zero neutron eDM, which reintroduces the problem.

8.1 The Axion as a Kaluza-Klein Zero Mode

Consider [27] a massless 5D scalar field $\phi(x^M)$, where $M = 0, 1, \ldots, 4$, with action

$$S_{5D} = \int d^5x\, \partial_M\phi\, \partial^M\phi. \tag{8.1}$$

Set the extra dimension $x^4 = y$ defining a circle of radius r with periodicity

$$y \equiv y + 2\pi r.$$

Our spacetime is now $M^4 \times S^1$. Periodicity in the y direction implies a discrete Fourier expansion:

$$\phi(x^\mu, y) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu)\, \exp\left(\frac{iny}{r}\right). \tag{8.2}$$

Notice that the Fourier coefficients $\phi_n(x^\mu)$ are functions of the standard 4D coordinates and therefore represent an infinite number of 4D scalar fields. The equations of motion for the Fourier modes are (in general massive) wave equations:

$$\partial_M \partial^M \phi = 0$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \left(\partial_\mu \partial^\mu - \frac{n^2}{r^2}\right)\phi_n(x^\mu) \exp\left(\frac{iny}{r}\right) = 0 \tag{8.3}$$

$$\Rightarrow \quad \partial_\mu \partial^\mu \phi_n(x^\mu) - \frac{n^2}{r^2}\phi_n(x^\mu) = 0. \tag{8.4}$$

These are then an infinite number of Klein–Gordon equations for massive 4D fields. This means that each Fourier mode ϕ_n is a 4D particle with mass

$$m_n^2 = \frac{n^2}{r^2}. \tag{8.5}$$

Only the zero mode ($n = 0$) is massless. One can visualize the states as an infinite tower of massive states (with increasing mass proportional to n). This is called the **Kaluza–Klein tower**, and the massive states ($n \neq 0$) are called *Kaluza–Klein* or *momentum states*, since they come from the momentum in the extra dimension:

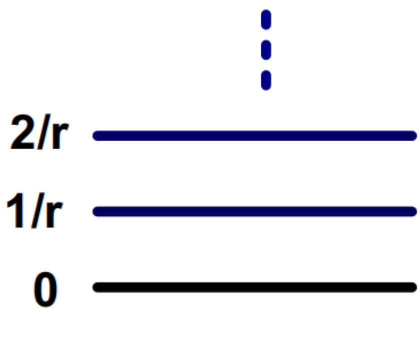

Figure 15: The Kaluza–Klein tower of massive states due to an extra S^1 dimension. Masses $m_n = |n|/r$ grow linearly with the fifth dimension's wave number $n \in \mathbb{Z}$.

In order to obtain the effective action in 4D for all these particles, let us plug the mode expansion of ϕ into the original 5D action:

$$S_{5D} = \int d^4x \int dy \sum_{n=-\infty}^{\infty} \left[\partial_\mu \phi_n(x^\mu)\, \partial^\mu \phi_n^*(x^\mu) - \frac{n^2}{r^2}|\phi_n|^2\right] \tag{8.6}$$

$$= 2\pi r \int d^4x\, \partial_\mu \phi_0(x^\mu)\, \partial^\mu \phi_0^*(x^\mu) + \ldots \tag{8.7}$$

$$= S_{4D} + \ldots \tag{8.8}$$

This means that the 5D action reduces to one 4D action for a massless scalar field plus an infinite sum of massive scalar actions in 4D. If we are only interested in energies smaller than the $1/r$ scale, we may concentrate only on the $n = 0$ mode action.

If we restrict our attention to the zero mode (like Kaluza did), then

$$\phi(x^M) = \phi(x^\mu).$$

This would be equivalent to truncating all the massive fields. In this case we speak of **dimensional reduction**. More generally, if we keep all the massive modes we talk about **compactification**, meaning that the extra dimension is compact and its existence is taken into account as long as the Fourier modes are included.

Therefore, a Kaluza-Klein (KK) zero mode is the lowest-energy mode of a field when compactified from higher dimensions to four dimensions. It is characterized by having no dependence on the extra-dimensional coordinates, meaning it is uniform along the compactified dimensions.

Dimensional Reduction[21]

Now, after expanding the previous 5D action, we get terms of the form

$$S^{5D} = \int \left[-\tfrac{1}{2g_5^2}\, dC \wedge \star dC \; - \; \tfrac{1}{2e_5^2}\, G \wedge \star G \; + \; \frac{N}{8\pi^2}\, C \wedge \mathrm{Tr}(G \wedge G) \right]. \tag{8.9}$$

Where g_5 and e_5 are the 5D gauge couplings of C and A_G respectively, with mass dimension $[g_5] = [e_5] = \frac{-1}{2}$, and $\mathbb{N} \in \mathbb{Z}$. Here, we denote the field strength of our U(1) and SU(3) gauge fields by dC and G, respectively. We adopt a minimal extra-dimensional axion model consisting of a one-form U(1) gauge field C and a one-form SU(3) gauge field C existing on an $\mathcal{R}^{1,3} \times S^1$ spacetime.

To derive the standard 4D axion-gluon action by dimensional reduction we previously consider a 5D theory of a U(1) gauge field C compactified to 4D on a circle S^1 with coordinate y $\equiv$ y+ 2πR. Later we identify the 4D axion θ as a Kaluza-Klein zero mode of C. That is:

$$\theta(x) = \int_{S^1} C(x) = \int_0^{2\pi R} C_5(x, y) dy \tag{8.10}$$

Hence, we get the ansatz

$$C(x, y) \;=\; \frac{1}{2\pi R}\, \theta(x)\, dy \;+\; (\text{massive KK modes}). \tag{8.11}$$

The factor $2\pi R$ is a convenient normalization so that θ has the usual 2π periodicity. Now, we can compute dC for the zero mode:

$$dC \;=\; \frac{1}{2\pi R}\, d\theta(x) \wedge dy. \tag{8.12}$$

The 1st term: Kinetic term

Plugging into the first kinetic piece,

$$S^C_{\text{kin}} = -\frac{1}{2g_5^2} \int_{M_4 \times S^1} dC \wedge *_5 dC = -\frac{1}{2g_5^2} \int_{M_4 \times S^1} \frac{1}{(2\pi R)^2}\, (d\theta \wedge dy) \wedge *_5 (d\theta \wedge dy). \tag{8.13}$$

Using the identity for the Hodge star when one factor is along the circle,

$$*_5(d\theta \wedge dy) \;=\; *_4 d\theta,$$

so

$$(d\theta \wedge dy) \wedge *_5 (d\theta \wedge dy) \;=\; dy \wedge (d\theta \wedge *_4 d\theta).$$

Integrating over the circle ($\int_{S^1} dy = 2\pi R$),

$$S^C_{\text{kin}} = -\frac{1}{2g_5^2}\frac{1}{(2\pi R)^2}(2\pi R)\int_{M_4} d\theta \wedge *_4 d\theta.$$

Thus,

$$S^C_{\text{kin}} = -\frac{1}{2}\left(\frac{1}{g_5^2\, 2\pi R}\right)\int_{M_4} d\theta \wedge *_4 d\theta.$$

We define the 4D axion decay constant f by

$$f^2 \equiv \frac{1}{g_5^2(2\pi R)}$$

so that the 4D kinetic term becomes

$$S^{\text{axion}}_{\text{kin}} = -\frac{1}{2}f^2\int_{M_4} d\theta \wedge * d\theta.$$

The 2nd term: Gauge Kinetic term

Similarly, we can reduce the gauge kinetic term. It arises because the gluon fields propagate in the extra dimensions. For the gauge 2-form G (assumed independent of y for the zero mode),

$$S_{\text{gauge}} = -\frac{1}{2e_5^2}\int_{M_4\times S^1} \mathrm{Tr}\, G \wedge *_5 G.$$

Since $*_5 G = *_4 G$ for G purely along M_4 directions,

$$S_{\text{gauge}} = -\frac{1}{2e_5^2}(2\pi R)\int_{M_4} \mathrm{Tr}\, G \wedge *_4 G.$$

Define the 4D gauge coupling e by

$$\frac{1}{e_4^2} \equiv \frac{2\pi R}{e_5^2} \qquad \Longleftrightarrow \qquad e_4^2 = \frac{e_5^2}{2\pi R},$$

so that

$$S_{\text{gauge}} = -\frac{1}{2e_4^2}\int_{M_4} \mathrm{Tr}\, G \wedge * G.$$

The 3rd term:Topological term

There are SU(3) gluon fields G also propagating in 5D. Hence, we obtain

$$\begin{aligned}\frac{N}{8\pi^2}\int_{M_4\times S^1} C\wedge \mathrm{Tr}[G\wedge G] &= \frac{N}{8\pi^2}\int_{M_4\times S^1}\frac{\theta(x)}{2\pi R}\,dy\wedge \mathrm{Tr}[G\wedge G]\\ &= \frac{N}{8\pi^2}\int_{M_4}\theta(x)\,\mathrm{Tr}[G\wedge G].\end{aligned}$$

Thus, the 4D form is proportional to the so-called Cher-Simons term.

$$\theta\,\mathrm{Tr}[G\wedge G].$$

Therefore, the final 4D zero-mode action is

$$S^{4D} = \int_{M_4}\left(-\frac{1}{2}f^2\,d\theta\wedge * d\theta - \frac{1}{2e_4^2}\,\mathrm{Tr}[G\wedge * G] + \frac{N}{8\pi^2}\,\theta\,\mathrm{Tr}[G\wedge G] + \cdots\right).$$

8.2 Associated Higher-Form Symmetries and their Dimensional Reduction

For now, we can consider the higher-form symmetries of the free 5D gauge field C and ignore the gluon coupling. The free gauge field C has the electric one-form symmetry $U(1)_e^{(1)}$ and the magnetic two-form symmetry $U(1)_m^{(2)}$. Both symmetries are continuous and have currents J_e and J_m, respectively, given by

$$J_e = \frac{1}{g_5^2} \star dC, \quad J_m = \frac{1}{2\pi} dC$$

The electric current J_e is conserved due to the 5D equations of motion $\mathrm{d}J_e \propto \mathrm{d}\star\ \mathrm{dC} = 0$, while the magnetic current J_m is conserved due to the Bianchi identity $\mathrm{d}J_m \propto \mathrm{ddC} = 0$.

Fate of these two symmetries upon dimensional reduction.

Electric one-form symmetry

In 5D, the transformation associated with the electric one-form symmetry is a shift of the gauge field C $\to$ C+ Λ where Λ is a flat connection. The $\mathcal{R}_{1,3}$ piece of this shift clearly yields an identical 4D one-form symmetry for the 4D gauge field remnant of C, provided this remnant has not been set to zero. Crucially, the S^1 piece of the symmetry gives rise to the continuous axion shift symmetry in 4D. This can be seen by simply plugging the shift into the definition of the axion (2) and carrying out the integral Under the 5D gauge transformation

$$C \ \mapsto \ C + d\Lambda \qquad (\Lambda \text{ a scalar}),$$
$$\theta \ \equiv \ \oint_{S^1} C.$$

The gauge field transforms as

$$\theta \ \mapsto \ \theta + \oint_{S^1} d\Lambda = \theta + \Big[\Lambda(2\pi R) - \Lambda(0)\Big] = \theta + \text{constant}$$

If Λ can jump by $2\pi n$ around the circle (a large gauge transformation), then θ shifts by $2\pi n$. This is why θ is a periodic variable, usually taken modulo 2π. Alternatively, the connection between the one-form symmetry and axion shift symmetry can be seen by the following calculation as

$$\theta(x) = \int_0^{2\pi R} C_y(x, y)\, dy.$$

Keeping only the zero mode (no y-dependence):

$$\theta(x) = (2\pi R)\, C_y(x).$$

So:

$$C_y(x) = \frac{1}{2\pi R}\, \theta(x).$$

Therefore,

$$\star_5 dC = \frac{1}{2\pi R} \star_4 d\theta.$$

Now, the definition of 5D current is

$$J_e = \frac{1}{g_5^2} \star_5 dC.$$

Upon reduction:

$$J_e \longrightarrow \frac{1}{g_5^2(2\pi R)} \star_4 d\theta.$$

Defining the axion decay constant as

$$f^2 \equiv \frac{1}{g_5^2(2\pi R)}.$$

So the 4D current becomes

$$J_s = f^2 \star_4 d\theta.$$

Magnetic two-form symmetry

The $5D$ magnetic $U(1)^{(2)}$ two-form symmetry dimensionally reduces to a magnetic one-form symmetry for the $4D$ remnant gauge field (provided it has not been set to zero by boundary conditions), and to a two-form winding symmetry for the axion.

This can be seen by dualizing C to a two-form field $\tilde{C}$. The magnetic two-form symmetry shifts $\tilde{C}$ by a flat connection:

$$\tilde{C} \rightarrow \tilde{C} + \tilde{\Lambda},$$

where Λ is a flat two-form. This amounts to a flat connection shift for the $4D$ remnant of $\tilde{C}$ and a two-form shift for the dual axion.
Alternatively, this can be seen directly from dimensional reduction: the $5D$ magnetic current is previously defined as

$$J_m = \frac{1}{2\pi} dC$$

Therefore,

$$J_w = \frac{1}{2\pi} \frac{(d\theta)}{(2\pi R)}$$

Hence, the continuous axion zero-form shift symmetry has its higher-dimensional origin in the electric $U^{(1)}(1)$ e one-form symmetry of C.
This is a crucial observation: the 4D axion potential is protected by the continuous axion shift symmetry. Since this shift symmetry is embedded in a one-form electric symmetry in extra-dimensional models, the quality of the axion can be understood in terms of the breaking and gauging of this electric one-form symmetry.
For the axion to acquire a potential, the shift symmetry and hence the electric one-form symmetry of C must be broken.

Appendices

A Ward Identity

For a complex scalar field, the Lagrangian is given by

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi \tag{W1}$$

Hence, the action is defined to be,

$$S[\phi, \phi^\dagger] = \int d^4x \, \mathcal{L} = \int d^4x \, \Big[(\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi \Big] \tag{W2}$$

If we consider a bunch of fields ϕ^a, then putting the action into the path integral we get:

$$Z = \int [\mathcal{D}\phi^a] \, e^{iS[\phi^a]} \tag{W3}$$

Now, if we shift the field and define a change of variables of the type:

$$\phi^a(x) \to \phi'^a(x) = \phi^a(x) + \epsilon(x) \delta \phi^a(x) \tag{W4}$$

We argue that this doesn't alter the numerical value of the functional integral:

$$\int [\mathcal{D}\phi^a] \, \exp(iS[\phi^a]) = \int [\mathcal{D}\phi'^a] \, \exp(iS[\phi'^a]) \tag{W5}$$

Because this is just a change of variables, and it is always true whether or not it is a symmetry operation.

Also, by discretizing the measure, we can see that the measure of the path usually doesn't change for a continuous symmetry operation, and we can write:

$$[\mathcal{D}\phi^a] = [\mathcal{D}\phi'^a] \tag{W6}$$

Here the Jacobian is ≈ 1. But as the variation parameter ϵ is not a constant, so the action won't remain invariant. The variation of the action due to the variation of the Lagrangian must be proportional to a derivative (to first order in $\epsilon(x)$), i.e.:

$$\delta S[\phi^a] = \int d^4x \, j^\mu(x) \partial_\mu \epsilon(x) = - \int d^4x \, \partial_\mu j^\mu(x) \epsilon(x) + \text{boundary term} \tag{W7}$$

$$\therefore S[\phi'^a] = S[\phi^a] - \int d^4x \, \partial_\mu j^\mu(x) \, \epsilon(x) \tag{W8}$$

Using (W8) into (W5) yields,

$$\int [\mathcal{D}\phi^a] \, e^{iS[\phi^a]} = \int [\mathcal{D}\phi^a] \, e^{iS[\phi'^a]} = \int [\mathcal{D}\phi^a] \, e^{i \, \left\{ S[\phi^a] - \int d^4x \, \partial_\mu j^\mu(x) \, \epsilon(x) \, \right\}}$$

$$\Rightarrow \int [\mathcal{D}\phi^a] \, e^{iS[\phi'^a]} = \int [\mathcal{D}\phi^a] \, e^{iS[\phi^a]} \Big(1 - i \int d^4x \, \partial_\mu j^\mu(x) \, \epsilon(x) + \mathcal{O}(\epsilon^2) \Big)$$

$$\therefore \int [D\phi^a] \, \Big(\int d^4x \, \partial_\mu j^\mu(x) \, \epsilon(x) \Big) \, e^{iS[\phi^a]} = 0 \tag{W9}$$

For (W9) to be true, we must have:

$$\partial_\mu j^\mu(x) = 0 \tag{W10}$$

Taking the expectation value with respect to vacuum state, we get:

$$\langle 0|\,\partial_\mu j^\mu(x)\,|0\rangle = 0 \tag{W11}$$

But it won't vanish in all states. To argue this, we can consider the 2-point function of the form:

$$\langle 0|T\phi^a(x_1)\phi^b(x_2)|0\rangle = \frac{1}{Z[0]}\int [\mathcal{D}\phi^a]\,\phi^a(x_1)\phi^b(x_2)\,e^{iS[\phi^a]} \tag{W12}$$

Under the change of variable $\phi^a \to \phi'^a$, the right side of equation (W12) is numerically equal to:

$$\frac{1}{Z[0]}\int [\mathcal{D}\phi'^a]\,\phi'^a(x_1)\phi'^b(x_2)\,e^{iS[\phi'^a]} \tag{W13}$$

As the correlation function is a physical result so it shouldn't change under the change of variables. Hence,

$$\begin{aligned}\frac{1}{Z[0]}\int [\mathcal{D}\phi^a]\,\phi^a(x_1)\phi^b(x_2)\,e^{iS[\phi^a]} &= \frac{1}{Z[0]}\int [\mathcal{D}\phi'^a]\,\phi'^a(x_1)\phi'^b(x_2)\,e^{iS[\phi'^a]}\\ &= \frac{1}{Z[0]}\int [\mathcal{D}\phi^a]\,\phi'^a(x_1)\phi'^b(x_2)\,e^{iS[\phi'^a]}\end{aligned} \tag{W14}$$

Now, plugging the change of variables and expanding the exponential we get:

$$\begin{aligned}&\frac{1}{Z[0]}\int [\mathcal{D}\phi^a]\,\phi^a(x_1)\phi^b(x_2)\,e^{iS[\phi^a]}\\ =&\frac{1}{Z[0]}\int [\mathcal{D}\phi^a]\left[\phi^a(x_1)\phi^b(x_2) + \epsilon(x_1)\delta\phi^a(x_1)\phi^b(x_2) + \phi^a(x_1)\epsilon(x_2)\delta\phi^b(x_2) + \mathcal{O}(\epsilon^2)\right]\\ &\qquad\qquad\qquad\qquad\times\left[1 - i\int d^4x\,\partial_\mu j^\mu(x)\,\epsilon(x)\right]e^{iS[\phi^a]}\end{aligned} \tag{W15}$$

The first term cancels on both sides, and neglecting higher-order terms in the variation, we obtain:

$$\begin{aligned}\frac{1}{Z[0]}\int [\mathcal{D}\phi]\Big(\epsilon(x_1)\delta\phi^a(x_1)\phi^b(x_2) + \epsilon(x_2)\phi^a(x_1)\delta\phi^b(x_2) - i\phi^a(x_1)\phi^b(x_2)\int d^4x, \partial_\mu j^\mu(x), \epsilon(x)\Big)\\ \times e^{iS[\phi]} = 0\end{aligned} \tag{W16}$$

Therefore, the integrand must vanish. Differentiating both sides and taking the expectation value of the integrands, we find:

$$i\langle\phi^b(x_1), \phi^c(x_2), \partial_\mu j^\mu(x)\rangle = \langle\delta\phi^b(x), \phi^c(x_2)\rangle, \delta^4(x - x_1) + \langle\phi^b(x_1), \delta\phi^c(x)\rangle, \delta^4(x - x_2) \tag{W17}$$

All expectation values here are time-ordered. Comparing this result with (W11), we observe that $\partial_\mu j^\mu$ is not conserved in the presence of operator insertions.
This outcome signifies that while computing correlation functions involving the divergence of the current and other operators charged under the symmetry, the divergence yields zero everywhere except at the operator insertion points. There, delta function contributions appear. This behavior becomes more evident when considering correlation functions with additional operator insertions.

This expression in (W17) is known as the Ward identity [28]. It is the quantum analog of Noether's theorem.

B Differential Forms

Differential forms [29] [30] (or just forms) are special class of tensors. A differential p-form is simply a (0, p) tensor that is completely antisymmetric. Thus, scalars are automatically 0-forms, and dual vectors are automatically 1-forms. We also have the 4-form $\epsilon_{\mu\nu\rho\sigma}$. The space of all p-forms is denoted Λ^p, and the space of all p-form fields on a manifold $\mathcal{M}$ is denoted $\Lambda^p(\mathcal{M})$.

Definition: p-form

A *p-form* ω on an n-dimensional manifold $\mathcal{M}$ is a totally antisymmetric covariant tensor field of rank p. It is something that can be integrated over p-dimensional space. In local coordinates $\{x^1, \ldots, x^n\}$, it can be written as:

$$\omega = \frac{1}{p!}\,\omega_{\mu_1\cdots\mu_p}(x)\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p} \tag{F1}$$

where $\omega_{\mu_1\cdots\mu_p}$ is an antisymmetric tensor field. Here, there are p factorial orders of differential terms and thats why, we normalize it by dividing p! in front of the term.

B.1 Wedge Product (Exterior Product)

Given a p-form A and a q-form B, we can form a $(p+q)$-form known as the **wedge product** $A \wedge B$ by taking the antisymmetrized tensor product:

$$(A\wedge B)_{\mu_1\ldots\mu_{p+q}} = \frac{(p+q)!}{p!\,q!}A_{[\mu_1\ldots\mu_p}B_{\mu_{p+1}\ldots\mu_{p+q}]} \tag{F2}$$

Thus, for example, the wedge product of two 1-forms is

$$(A\wedge B)_{\mu\nu} = 2A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu \tag{F3}$$

Note that

$$A\wedge B = (-1)^{pq}B\wedge A \tag{F4}$$

The **wedge product** $\wedge$ is an operation on differential forms that satisfies:

- Antisymmetry: $\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$,
- Nilpotency: $\mathrm{d}x^i \wedge \mathrm{d}x^i = 0$,

Examples

- **0-form**: Scalar function

$$f = f(x, y, z) \tag{F5}$$

- **1-form**:

$$\omega = A(x)\,\underline{\mathrm{d}x} + B(x)\,\underline{\mathrm{d}y} + C(x)\,\underline{\mathrm{d}z} \tag{F6}$$

1-forms are to be related to quantities that occur as the integrands of line integrals.

- **2-form**:

$$\omega = f(x)\,\underline{\mathrm{d}x \wedge \mathrm{d}y} + g(x)\,\underline{\mathrm{d}y \wedge \mathrm{d}z} \tag{F7}$$

2-forms are to be related to the quantites that occur as the integrads of surface integrals.

- **3-form**:

$$\omega = h(x)\,\underline{\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z} \tag{F8}$$

3-forms are to be related to quantities that occur as the integrans of volume integrals.

Summary Table

Form Type	Degree	Example
0-form	0	$f(x)$
1-form	1	$f(x)\,\mathrm{d}x$
2-form	2	$\frac{1}{2!}f(x)\,\mathrm{d}x \wedge \mathrm{d}y$
3-form	3	$\frac{1}{3!}f(x)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$
p-form	p	$\dfrac{1}{p!}\omega_{\mu_1\cdots\mu_p}\,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$

In Physics:

1. **Gauge Field**

In particle physics, we can integrate a gauge field over the world line of some charged particle. Hence, we can define an integral over this path of the quantity

$$\int_\gamma A_\mu\, dx^\mu = \int_\gamma A, \quad \text{i.e.} \quad A = A_\mu\, dx^\mu; \quad \text{A is just a dimensionless thing} \tag{F9}$$

So, the differential form is something that kind of packages together these tensor components with indices.

2. **Field strength**

$F_{\mu\nu}$ is a 2-form as it takes two vector indices and the differential form version is that packages together $F_{\mu\nu}$ with two different differentials dx^μ and dx^ν , i.e.

$$F = \frac{1}{2}F_{\mu\nu}\, dx^\mu \wedge dx^\nu \tag{F10}$$

And, we can integrate F over some 2-dimensional surface:

$$\int_\Sigma F \tag{F11}$$

where Σ is a 2-dimensional surface.

3. **Scalar kinetic term**

$$S = \int d^dx\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) \tag{F12}$$

$$= \frac{1}{2}\int d\phi \wedge \star\, d\phi \tag{F13}$$

4. **Gauge Field Kinetic Term**

$$S = \int d^d x \sqrt{-g} \left(-\frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} \right) \tag{F14}$$

$$= \int \left(-\frac{1}{2e^2} F \wedge \star F \right) \tag{F15}$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}; \quad \star F = \frac{1}{2} \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu; \quad \text{In } d = 4 \quad \tilde{F}^{\mu\nu} = \star F^{\mu\nu} \tag{F16}$$

5. **Theta Term**

$$S = \int d^4 x \sqrt{-g} \left(\frac{\theta}{16\pi^2} \right) \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{F17}$$

$$= \int \left(\frac{\theta}{8\pi^2} \right) \star F \wedge F \tag{F18}$$

B.2 Hodge Star Operator ($\star$)

It is also known as complementary Differential Forms. Associated with each differential form is a complementary (or dual) form that contains the differentials not included in the original form. Thus, if our underlying space has dimension d, the form dual to a p-form will be a $(d-p)$-form. In three dimensions, the complement to a 1-form will be a 2-form (and vice versa), while the complement to a 3-form will be a 0-form (a scalar). It is useful to work with these complementary forms, and this is done by introducing an operator known as the Hodge operator, it is usually designated notationally as an asterisk (preceding the quantity to which it is applied, not as a superscript), and is therefore also referred to either as the Hodge star operator or simply as the star operator. Formally, its definition requires the introduction of a metric and the selection of an orientation (chosen by specifying the standard order of the differentials comprising the 1-form basis).

Let (M, g) be an n-dimensional oriented Riemannian manifold.
The **Hodge star operator** is a linear map:

$$\star : \Omega^p(M) \to \Omega^{n-p}(M) \tag{F19}$$

defined by the condition that for all $\alpha, \beta \in \Omega^p(M)$,

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle \, \mathrm{vol}_g \tag{F20}$$

where:

- $\langle \alpha, \beta \rangle$ is the pointwise inner product induced by the metric g,
- vol_g is the volume form determined by g and the orientation of M.

Examples:

In Euclidean Space

the flat Euclidean metric is:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{F21}$$

Hence, the effect of Hodge star on the various possible differential forms are:

$$\begin{aligned} &\star 1 = dx \wedge dy \wedge dz \\ \star \mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z, \quad &\star \mathrm{d}y = \mathrm{d}z \wedge \mathrm{d}x, \quad \star \mathrm{d}z = \mathrm{d}x \wedge \mathrm{d}y \\ \star(dx \wedge dy) = dz \quad &\star (dz \wedge dx) = dy \quad \star (dy \wedge dz) = dx \\ &\star (dx \wedge dy \wedge dz) = 1 \end{aligned} \tag{F22}$$

In Minkowski Space

Taking the oriented 1-form basis (dt, dx_1, dx_2, dx_3) and the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{F23}$$

Let's now determine the effect of the Hodge operator on the various possible differential forms.

$$\star 1 = dt \wedge dx \wedge dy \wedge dz \tag{F24}$$

$$\star \mathrm{d}x = dt \wedge \mathrm{d}y \wedge \mathrm{d}z, \quad \star (dt \wedge dx) = -\mathrm{d}y \wedge \mathrm{d}z, \quad \star \mathrm{d}t = \mathrm{d}x \wedge \mathrm{d}y \wedge dz \tag{F25}$$

$$\star(dy \wedge dz) = dt \wedge dx \quad \star (dt \wedge dx) = -dy \wedge dz \quad \star (dt \wedge dx \wedge dy) = dz \tag{F26}$$

$$\star (dt \wedge dx \wedge dy \wedge dz) = -1 \tag{F27}$$

A requirement of dualities between vector spaces is that the original and transformed spaces have the same dimensionality; this is true of the spaces of p- and $(n-p)$-forms.
In local coordinates, the Hodge star is defined as:

$$\star\omega = \frac{\sqrt{|\det g|}}{p!(n-p)!} \, \omega_{i_1 \cdots i_p} \, \varepsilon^{i_1 \cdots i_p}{}_{j_{p+1} \cdots j_n} \, dx^{j_{p+1}} \wedge \cdots \wedge dx^{j_n}, \tag{F28}$$

where

$$\varepsilon^{i_1 \cdots i_p j_{p+1} \cdots j_n} = g^{j_{p+1} k_{p+1}} \cdots g^{j_n k_n} \, \varepsilon^{i_1 \cdots i_p k_{p+1} \cdots k_n} \tag{F29}$$

is obtained by lowering indices on the Levi–Civita symbol.

Examples

- When $p = 0$, a function $\omega \in \Omega^0(\mathbb{R}^3)$ satisfies:

$$\star\omega = \frac{1}{0!\,3!} \, \omega \, \varepsilon_{ijk} \, dx^i \wedge dx^j \wedge dx^k = \omega \, dx^1 \wedge dx^2 \wedge dx^3 \in \Omega^3(\mathbb{R}^3). \tag{F30}$$

- When $p = 3$, a top-form $\omega = \frac{1}{3!}\,\omega_{ijk}\,dx^i \wedge dx^j \wedge dx^k$ yields:

$$\star\omega = \frac{1}{3! \cdot 0!}\,\omega_{ijk}\,\varepsilon^{ijk} = \omega_{123} \in \Omega^0(\mathbb{R}^3), \tag{F31}$$

  recovering a function.

- The action of the Hodge star on the unit 0-form 1 yields the invariant volume form:

$$\star 1 = \sqrt{|g|}\,dx^1 \wedge \cdots \wedge dx^D. \tag{F32}$$

  More generally, if

$$\omega = \frac{1}{p!}\,\omega_{i_1 \cdots i_p}\,dx^{i_1} \wedge \cdots \wedge dx^{i_p} \in \Omega^p(M), \tag{F33}$$

  then its Hodge dual has components:

$$(\star\omega)_{j_1 \cdots j_{D-p}} = \frac{\sqrt{|g|}}{p!(D-p)!}\,\varepsilon_{j_1 \cdots j_{D-p}}{}^{i_1 \cdots i_p}\,\omega_{i_1 \cdots i_p}. \tag{F34}$$

- Applying the Hodge star twice gives:

$$\star(\star\omega) = (-1)^{p(D-p)}\omega \quad \text{(Riemannian)}, \qquad \star(\star\omega) = -(-1)^{p(D-p)}\omega \quad \text{(Lorentzian)}. \tag{F35}$$

Inner Product and Codifferential

Given two p-forms $\omega, \xi \in \Omega^p(M)$, their wedge with the Hodge dual of the second gives a top-form:

$$\omega \wedge \star\xi = \frac{1}{p!}\,\omega_{\mu_1 \cdots \mu_p}\,\xi^{\mu_1 \cdots \mu_p}\,\sqrt{|g|}\,dx^1 \wedge \cdots \wedge dx^D, \tag{F36}$$

which defines the inner product:

$$\langle \omega, \xi \rangle = \int_M \omega \wedge \star\xi = \int_M d^D x\,\sqrt{|g|}\,\omega_{\mu_1 \cdots \mu_p}\,\xi^{\mu_1 \cdots \mu_p}. \tag{F37}$$

The adjoint exterior derivative (codifferential) is defined by:

$$d^\dagger = (-1)^{Dp+D+1} \star d \star, \tag{F38}$$

so that for any $\eta \in \Omega^{p-1}(M)$ and $\omega \in \Omega^p(M)$, we have the integration by parts formula:

$$\langle d\eta, \omega \rangle = \langle \eta, d^\dagger \omega \rangle. \tag{F39}$$

A p-form is called:

- co-closed if $d^\dagger \omega = 0$
- co-exact if there exists $\beta \in \Omega^{p+1}(M)$ such that $\omega = d^\dagger \beta$

Maxwell's Equations and Noether Current in Differential Forms

Electromagnetic Field Strength 2-Form

$$F = \frac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{F40}$$

Maxwell's Equations

$$\mathrm{d}F = 0 \quad \text{(homogeneous equations)}$$
$$\mathrm{d} \star F = \star J \quad \text{(inhomogeneous equations)}$$

Explanation of Symbols

- F: Electromagnetic field strength 2-form, which encodes $\vec{E}$ and $\vec{B}$
- $\star$: Hodge dual operator, which maps p forms to (n-p) forms, depends on the metric.
- $J = J_\mu\, \mathrm{d}x^\mu$: Current 1-form
- dF = 0 : This captures Faraday's law and no magnetic monopoles.
- d $\star$ F = $\star$ J: This encodes Gauss's law and Ampere's law

Noether Current

From $U(1)$ gauge symmetry, the conserved Noether current j satisfies:

$$\mathrm{d} \star j = 0 \tag{F41}$$

This means $\star$ j is a closed 3-form, and hence j is conserved:

$$\partial_\mu j^\mu = 0 \tag{F42}$$

B.3 Differentiating Forms

Exterior Derivative

we define the exterior derivative, which we consider to be an operator identified by the traditional symbol d. We have, in fact, already introduced that operator when we wrote dx_i, stating at the time that we intended to interpret dx_i as a mathematical object with specified properties and not just as a small change in x_i . We are now refining that statement to interpret dx_i as the result of applying the operator d to the quantity x_i. The exterior derivative is a linear map such that

$$d : \Omega^r(M) \longrightarrow \Omega^{r+1}(M), \tag{F43}$$

The simplest example is the gradient, which is the exterior derivative of a 0-form:

$$(d\varphi)_\mu = \partial_\mu \varphi \tag{F44}$$

Exterior derivatives obey a modified version of the Leibniz rule when applied to the product of a p-form ω and a q-form η:

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge (d\eta) \tag{F45}$$

1. Exterior Derivative of a 0-form (Function)

Let $f(x, y, z) \in \Omega^0$, then

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z \tag{F46}$$

This is the **gradient**.

2. Exterior Derivative of a 1-form

Let $\alpha = f(x, y)\,\mathrm{d}x + g(x, y)\,\mathrm{d}y \in \Omega^1$, then

$$\mathrm{d}\alpha = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y \tag{F47}$$

This is the **curl** in 2D.

3. Exterior Derivative of a 2-form

In 3D, this corresponds to the **divergence**. Let

$$\beta = f(x, y, z)\,\mathrm{d}y \wedge \mathrm{d}z + g(x, y, z)\,\mathrm{d}z \wedge \mathrm{d}x + h(x, y, z)\,\mathrm{d}x \wedge \mathrm{d}y \tag{F48}$$

then

$$\mathrm{d}\beta = \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \tag{F49}$$

We complete our definition of the operator d by requiring it to have the following properties, where ω is a p-form, ω' is a p'-form, and f is an ordinary function (a 0-form):

$$d(\omega + \omega') = d\omega + d\omega' \quad (p = p') \tag{F50}$$

$$d(f\omega) = (df) \wedge \omega + f\,d\omega \tag{F51}$$

$$d(\omega \wedge \omega') = d\omega \wedge \omega' + (-1)^p \omega \wedge d\omega' \tag{F52}$$

$$df = \sum_j \frac{\partial f}{\partial x_j} dx_j \tag{F53}$$

Another interesting fact about exterior differentiation is that, for any form A,

$$d(dA) = 0 \tag{F54}$$

which is often written $d^2 = 0$. This identity is a consequence of the definition of d and the fact that partial derivatives commute, $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$ (acting on anything). This leads us to the following mathematical idea. We define a p-form A to be **closed** if $dA = 0$, and **exact** if $A = dB$ for some $(p - 1)$-form B. Obviously, all exact forms are closed, but the converse is not necessarily true.

B.4 Applications of Differential Forms in Gauge Theory

In various cases, we use differential forms to deal with field strengths and gauge fields, even at the level of the action. Let us focus on gauge fields.

p-form Field Strength

Consider a p-rank antisymmetric tensor $F^{(p)}_{\mu_1\cdots\mu_p}$. We can define the differential form:

$$F^{(p)} = \frac{1}{p!} F^{(p)}_{\mu_1\cdots\mu_p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}. \tag{F55}$$

Taking the exterior derivative:

$$dF^{(p)} = \frac{1}{p!} \partial_{\mu_0} F^{(p)}_{\mu_1\cdots\mu_p}\, dx^{\mu_0} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}, \tag{F56}$$

or more symmetrically:

$$dF^{(p)} = \frac{1}{(p+1)!} \Big(\partial_{\mu_0} F^{(p)}_{\mu_1\cdots\mu_p} + (-1)^p (\text{cyclic permutations of } \mu_0, \mu_1, \ldots, \mu_p) \Big)\, dx^{\mu_0} \wedge \cdots \wedge dx^{\mu_p} \tag{F57}$$

Example: 2-form

Taking the exterior derivative of a 2-form:

$$F^{(2)} = \frac{1}{2} F^{(2)}_{\mu_1\mu_2}\, dx^{\mu_1} \wedge dx^{\mu_2}, \tag{F58}$$

we find:

$$dF^{(2)} = \frac{1}{2!} \partial_{\mu_0} F^{(2)}_{\mu_1\mu_2}\, dx^{\mu_0} \wedge dx^{\mu_1} \wedge dx^{\mu_2} \tag{F59}$$

which agrees with the general expression:

$$dF^{(2)} = \frac{1}{3!} \Big(\partial_{\mu_0} F^{(2)}_{\mu_1\mu_2} + \partial_{\mu_1} F^{(2)}_{\mu_2\mu_0} + \partial_{\mu_2} F^{(2)}_{\mu_0\mu_1} \Big)\, dx^{\mu_0} \wedge dx^{\mu_1} \wedge dx^{\mu_2} \tag{F60}$$

Levi–Civita Symbol and Tensor

In d dimensions, the totally antisymmetric Levi–Civita symbol $\epsilon_{\mu_1\cdots\mu_d}$ is normalized by:

$$\epsilon_{01\cdots d} = +1. \tag{F61}$$

With the Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$, indices are raised trivially:

$$\epsilon^{\mu_0\cdots\mu_d} = \epsilon_{\mu_0\cdots\mu_d}. \tag{F62}$$

The Levi–Civita tensor is defined as:

$$\varepsilon_{\mu_0\cdots\mu_d} = \sqrt{-g}\, \epsilon_{\mu_0\cdots\mu_d}, \tag{F63}$$

$$\varepsilon^{\mu_0\cdots\mu_d} = \sqrt{-g}\, g^{\mu_0\nu_0} \cdots g^{\mu_d\nu_d} \epsilon_{\nu_0\cdots\nu_d} = \operatorname{sgn}(g) \sqrt{-g}\, \epsilon^{\mu_0\cdots\mu_d}. \tag{F64}$$

Hodge Dual of a p-form

Given a p-form:

$$F^{(p)} = \frac{1}{p!} F_{\nu_1\cdots\nu_p}\, dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_p}, \tag{F65}$$

its Hodge dual $\star F^{(p)} \in \Omega^{d-p}(M)$ has components:

$$(\star F^{(p)})_{\mu_1\cdots\mu_{d-p}} = \frac{1}{p!} \varepsilon_{\mu_1\cdots\mu_{d-p}\nu_1\cdots\nu_p}\, F^{\nu_1\cdots\nu_p} \tag{F66}$$

Wedge Product and Hodge Dual

Given two forms $F^{(p)}$ and $F^{(q)}$, the wedge product with the Hodge dual gives:

$$F^{(p)} \wedge \star F^{(q)} = \frac{1}{p!(D-q)!} F_{\mu_1\cdots\mu_p} (\star F^{(q)})_{\sigma_1\cdots\sigma_{D-q}} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \wedge dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_{D-q}} \tag{F67}$$

$$= \frac{1}{p!q!(D-q)!} \varepsilon_{\sigma_1\cdots\sigma_{D-q}\nu_1\cdots\nu_q} F^{(p)}_{\mu_1\cdots\mu_p} F^{(q)}_{\nu_1\cdots\nu_q} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \wedge dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_{D-q}} \tag{F68}$$

Dual of the Dual

The dual of a dual field strength is:

$$\star(\star F^{(p)}) = \mathrm{sgn}(g)(-1)^{p(D-p)} F^{(p)} = -(-1)^{p(D-p)} F^{(p)}, \tag{F69}$$

depending on the metric signature (Lorentzian here).

Summary

A	1-form	Electromagnetic potential
$F = dA$	2-form	Field strength
J	1-form	Charge-current 4-vector
$\star J$	3-form	Dual of current
$dF = 0$	3-form	Homogeneous Maxwell equations, (Bianchi identity)
$d \star F = \star J$	3-form	Inhomogeneous Maxwell equations
$d(\star J) = 0$	4-form	Continuity equation (charge conservation)

B.5 Integrating Forms

Stokes' Theorem

A key result regarding the integration of differential forms is a formula known as Stokes' theorem which in its simplest form, states that if -

- R is a simply-connected region (i.e., one with no holes) of a p-dimensional differentiable manifold in a n-dimensional space ($n \geq p$);
- R has a boundary denoted ∂R, of dimension $p-1$;
- ω is a $(p-1)$ form defined on R and its boundary, with derivative $d\omega$;

then,

$$\int_R d\omega = \int_{\partial R} \omega \tag{F70}$$

As the $d\omega$ results from applying the d operator to ω, the differentials in $d\omega$ consist of all those in ω, in the same order, but preceded by that produced by the differentiation.
This equates a boundary integral with a bulk integral involving a derivative.

Submanifold

A **submanifold** is a lower-dimensional "surface" embedded in a higher-dimensional manifold. For example:

- A curve in 3D space is a 1-dimensional submanifold.
- A surface (like a sphere or plane) in 3D is a 2-dimensional submanifold.
- A point is a 0-dimensional submanifold.

Formally, a submanifold $M^k \subset X^n$ has dimension $k < n$, and it inherits geometric properties from the ambient manifold X.

Role in Differential Forms

Differential forms are naturally integrated over submanifolds:

A k-form ω can be integrated over a k-dimensional submanifold C_k:

$$\int_{C_k} \omega \tag{F71}$$

This is the geometric generalization of line integrals (for 1-forms), surface integrals (for 2-forms), and so on.

Example

Let

$$\omega = f(x)\, dx \wedge dy, \tag{F72}$$

a 2-form in $\mathbb{R}^3$. Then ω can be integrated over a 2D surface $S \subset \mathbb{R}^3$ to yield:

$$\int_S f(x)\, dx \wedge dy. \tag{F73}$$

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To Whom It May Concern

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