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Internship Report

on

Properties of Scattering Amplitudes

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Approval

The internship report titled "**Properties of Scattering Amplitudes**" submitted by **Mohammad Tahmid Sahriar**, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfilment of the requirements of the internship program.

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1 Introduction

In physics, we try to understand how nature works at different scales. At very small scales, such as atoms or subatomic particles, classical physics no longer gives the right answers. This is where quantum mechanics comes in. It helps us describe systems where the number of particles is fixed, like electrons in an atom.

However, in high-energy physics, things become even more interesting. When particles collide at very high energies, they can create or destroy other particles. In such cases, the number of particles is not conserved. To study these situations, we need a more powerful theory called **Quantum Field Theory (QFT)**.

In QFT, every interaction between particles is treated as a *scattering process*. These processes are not deterministic, meaning we cannot predict exactly what will happen in a single event. Instead, we calculate the *probability amplitudes* of different outcomes. These amplitudes are found using something called the **S-matrix**, which connects the initial and final states of a system.

Most quantum field theories are too complicated to solve exactly. Therefore, we use an approach called *perturbation theory*, where we expand the interaction in small steps (in powers of the coupling constant). Each step can be represented visually by a **Feynman diagram**. By applying certain mathematical rules, known as **Feynman rules**, we can calculate the contribution of each diagram to the total scattering amplitude.

Often, these diagrams include internal or *virtual particles* whose momenta are not fixed by the external ones. To include all possibilities, we integrate over their momenta, leading to what we call **Feynman integrals**. These integrals are usually complex and can even diverge (become infinite) at certain points. To handle this, we study them as functions of complex momenta and use a process called **analytic continuation** to extend the results to regions where they make sense physically.

Understanding where these integrals become undefined or singular is very important. The conditions for these singularities are described by the **Landau equations**. During this internship, I studied a classic paper that provides an analytic proof of these singularities and explores how these ideas extend to complex singularities. This study helped me appreciate how deep and mathematically rich the structure of quantum field theory really is.

2 Feynman Integrals

In this section, I briefly explain how Feynman integrals arise from Feynman diagrams and how we simplify them in practice.

2.1 Building blocks of a diagram

A Feynman diagram has two basic ingredients:

- **Lines:** Each line stands for a particle. External lines represent incoming or outgoing particles, while internal lines represent *virtual* particles.
- **Vertices:** Points where lines meet. Each vertex represents an interaction, and energy–momentum is conserved at every vertex.

For a given scattering process, there can be many diagrams: tree-level, one-loop, two-loop, and so on. Diagrams can also be planar or non-planar. In principle, to compute a scattering amplitude, we must sum the contributions from all allowed diagrams.

2.2 From diagrams to integrals: the rules

The Feynman rules tell us how to translate a diagram into a mathematical expression:

1. **Assign loop momenta.** Give a momentum k_j to each independent loop. By choosing a convenient set of closed loops, we get a set of independent loop momenta.
2. **Use conservation at each vertex.** Express every internal (virtual) momentum q_i in terms of the loop momenta k_j and the external momenta p_k. Each q_i becomes a linear function of $\{k_j\}$ and $\{p_k\}$.
3. **Propagators for internal lines.** Each internal line contributes a factor
$$\frac{i}{q_i^2 - m_i^2 + i\epsilon}.$$
4. **Integrate over loop momenta.** For each loop momentum k_j, integrate with the Lorentz-invariant measure
$$\int \frac{d^4 k_j}{(2\pi)^4}.$$

After applying these steps to all loops, the full expression is a multiple integral over the loop momenta. Typically, the denominator contains a product of many propagator factors, which makes direct evaluation difficult.

2.3 Feynman parametrization

To combine multiple denominators into a single one, we use *Feynman parametrization*. The basic two-denominator identity is

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{\big[xA + (1-x)B\big]^2} \,, \tag{1}$$

which already simplifies many integrals.

For N denominators, the general form (up to overall constants that do not affect the discussion here) is

$$\frac{1}{A_1 A_2 \cdots A_N} \propto \int_0^1 \cdots \int_0^1 \left(\prod_{i=1}^N d\alpha_i \right) \delta\left(1 - \sum_{i=1}^N \alpha_i\right) \frac{1}{\left[\sum_{i=1}^N \alpha_i A_i\right]^N} \,. \tag{2}$$

Here, the α_i are the *Feynman parameters* and the delta function enforces $\sum_i \alpha_i = 1$.

2.4 General structure of a Feynman integral

After introducing Feynman parameters and shifting the loop momenta as needed, a typical m-loop integral with N propagators takes the schematic form

$$\int_0^1 \cdots \int_0^1 \left(\prod_{i=1}^N d\alpha_i \right) \delta\left(1 - \sum_{i=1}^N \alpha_i\right) \int \frac{d^4 k_1}{(2\pi)^4} \cdots \int \frac{d^4 k_m}{(2\pi)^4} \frac{1}{\big[F(\alpha, k, p) + i\epsilon\big]^n} \,, \tag{3}$$

where,

$$F(\alpha, k, p) = \sum_{i=1}^N \alpha_i \big(q_i^2 - m_i^2\big) \,. \tag{4}$$

The function F depends on the Feynman parameters, the loop momenta, and the external momenta. After the loop integrations are performed, the result becomes a function of the external kinematics alone. In many interesting cases, these integrals develop divergences or singularities.

3 Singularities of Integral Representations

As mentioned in the introduction, to understand the analytic structure of Feynman integrals it is essential to know where their singularities occur. When a function is given by an integral, possible singularities of the function are tightly connected to how the singularities of the *integrand* move relative to the integration contour. In this section, following the simple ideas outlined by Polkinghorne and Screaton (1960), I start with one-variable contour integrals and then indicate how the picture extends to several variables and multiple integrals.

3.1 Singularities of simple integrals

Let $g(z, w)$ be analytic in two complex variables, and let C be a finite contour in the complex w-plane. Define

$$f(z) = \int_C g(z, w)\, dw\,. \tag{5}$$

Suppose we know the singularities of the integrand with respect to w and that, for each fixed z, their locations are

$$w = w_r(z), \qquad r = 1, 2, \ldots\,. \tag{6}$$

Assume there is a neighborhood of some point z_0 in which the contour C is free of these singularities $w_r(z)$. Then the integral in (5) is well-defined and $f(z)$ is analytic at z_0.

We now ask: what happens when we analytically continue z away from z_0? As z moves, the points $w_r(z)$ move in the w-plane. The value of the integral (5) remains unchanged as long as no singularity crosses the contour C. In fact, by the Cauchy–Goursat theorem, if C' is any other contour with the same endpoints as C and if the region between C and C' contains no singularities, then

$$\int_C g(z, w)\, dw = \int_{C'} g(z, w)\, dw\,. \tag{7}$$

This means we can *deform* the contour continuously to avoid moving singularities and thus continue $f(z)$ analytically along a path in the z-plane.

3.1.1 When can singularities of $f(z)$ arise?

The deformation procedure can fail in only a few characteristic ways. These failures signal singularities of $f(z)$:

(i) **End-point singularities.** If a singularity $w_r(z)$ hits an endpoint of the contour, say A or B, no small deformation keeping the endpoints fixed can avoid it. Then the integral (5) becomes ill-defined and z reaches a singularity of f.

(ii) **Pinching singularities.** If two (or more) singularities approach the contour from opposite sides and meet on it, the contour is *pinched* and cannot be deformed away. A related situation is when one singularity is fixed and another approaches it with the contour in between. Pinching requires approach from opposite sides of C.

(iii) **Infinite deformations.** If in trying to avoid a moving singularity the contour is dragged off to infinity, the integral may diverge because the contour is no longer finite. By a change of variables such as $w = 1/\zeta$, the point at infinity is brought to a finite point, which reduces this case to a variant of the pinching scenario in (ii).

3.2 A compact lemma (after Polkinghorne–Screaton)

Lemma 1 (Contour-deformation criterion)**.** *Let $f(z)$ be defined by* (5) *with singular sets $\{w_r(z)\}$ as in* (6)*. In any simply connected domain of z where the contour can be deformed between its endpoints to avoid all $w_r(z)$, $f(z)$ admits an analytic continuation. Singularities of $f(z)$ can occur only when one of the following obstructions arises: (i) an end-point singularity, (ii) a pinching of the contour by singularities approaching from opposite sides, or (iii) an infinite deformation (equivalently, a pinching after a map that brings infinity to a finite point).*

3.3 Beyond one variable

For several complex variables and multiple integrals, the same ideas persist: the singular set of the *integrand* moves in a higher-dimensional space and the allowed deformations of the integration domain determine whether the integral defines an analytic function of external parameters. In practical applications to Feynman integrals, this geometric picture is the seed for the Landau analysis of singularities, where the *pinch* conditions become the Landau equations.

4 Landau Singularities

In the study of Feynman integrals, one of the key questions is to understand where and why these integrals become singular or undefined. These singularities are not just mathematical accidents; they have deep physical meanings. They usually signal the points where new physical processes become possible, such as the production of new particles. The analysis of such singularities was first carried out systematically by the physicist Lev Landau in 1959, and the conditions he derived are now known as the **Landau equations**.

4.1 Physical idea behind the singularities

Each Feynman diagram corresponds to an integral over internal momenta. The denominators in such an integral have the general form

$$\frac{1}{q_i^2 - m_i^2 + i\epsilon}, \tag{8}$$

where q_i is the four-momentum of an internal (virtual) particle and m_i is its mass. When any of these denominators vanish, that is, when

$$q_i^2 - m_i^2 = 0\,, \tag{9}$$

the corresponding internal particle can *go on-shell*, meaning it can behave like a real particle. These are the points where the Feynman integral may develop a singularity.

However, not every time a denominator goes to zero do we get a singularity. The integral only becomes singular when these vanishing denominators coincide in such a way that the contour of integration in momentum space cannot be deformed away from them. This situation is very similar to the *pinching singularities* we discussed earlier for one-dimensional integrals, but now it happens in many dimensions.

4.2 The basic structure of Landau's argument

Landau considered the Feynman integral in its parameterized form,

$$I(p) = \int \frac{d^4k_1\, d^4k_2 \cdots d^4k_L}{(q_1^2 - m_1^2 + i\epsilon)(q_2^2 - m_2^2 + i\epsilon)\cdots(q_N^2 - m_N^2 + i\epsilon)}\,, \tag{10}$$

where p represents the external momenta, L is the number of loops, and N is the number of propagators. Using Feynman parameters α_i, the integral can be expressed in a more symmetric form where all denominators combine into one overall factor.

Landau showed that the singularities of the integral occur at points where the integration contour in momentum space becomes **pinched** between singularities of the integrand. The conditions for such pinching give rise to the following equations, now known as the Landau equations.

4.3 Landau equations

The Landau equations provide a set of necessary conditions for the occurrence of singularities in Feynman integrals:

1. **On-shell conditions:**

$$q_i^2 - m_i^2 = 0 \qquad \text{for all } i \text{ with } \alpha_i \neq 0\,. \tag{11}$$

These conditions state that the internal lines corresponding to nonzero Feynman parameters α_i must go on-shell, meaning they can represent real particles.

2. **Momentum-balance (stationary) conditions:**

$$\sum_{i=1}^{N} \alpha_i q_i = 0\,. \tag{12}$$

This ensures that the combination of momenta along each independent loop balances in such a way that the integration contour cannot be deformed without crossing a singularity.

When these two sets of conditions are satisfied simultaneously, the integral in Eq. (10) may develop a singularity. The points in momentum space where this happens are called **Landau singularities**.

Together, these two sets of equations define the locations of possible singularities in terms of the external momenta and internal masses. The solutions of the Landau equations correspond to the boundaries in momentum space where the analytic structure of the scattering amplitude changes. In physical terms, these are the threshold points where new real processes, such as particle creation, can begin to occur. Thus, the Landau equations provide a direct and elegant connection between the geometry of Feynman diagrams and the analytic behavior of quantum field theory amplitudes.

5 Application to the Triangle Graph

As an application of the Landau equations, let us consider the case of a simple **triangle diagram**. We aim to determine the condition for which the Feynman integral corresponding to this graph develops a singularity. In particular, we will find the equation of the hypersurface in terms of the external momenta that may contain these singularities.

For simplicity, we take all internal and external lines to have the same mass m, and we assume that all external momenta are incoming. We also neglect cases where any Feynman parameter α_i vanishes, and we do not impose the on-shell condition on the external legs.

5.1 Setup and conventions

For the triangle diagram, let all external momenta be incoming, so that

$$p_1 + p_2 + p_3 = 0\,. \tag{13}$$

We denote the internal momenta of the three propagators by

$$q_1 = k, \qquad q_2 = k + p_1, \qquad q_3 = k - p_3\,, \tag{14}$$

where k is the loop momentum. Each internal line carries the same mass m.

5.1.1 Applying the Landau equations

The first set of Landau equations (on-shell conditions) for the three internal lines give:

$$q_1^2 - m^2 = 0 \quad \Rightarrow \quad k^2 = m^2\,, \tag{15}$$

$$q_2^2 - m^2 = 0 \quad \Rightarrow \quad (k+p_1)^2 = m^2 \quad \Rightarrow \quad 2kp_1 + p_1^2 = 0, \tag{16}$$

$$q_3^2 - m^2 = 0 \quad \Rightarrow \quad (k-p_3)^2 = m^2 \quad \Rightarrow \quad -2kp_3 + p_3^2 = 0. \tag{17}$$

5.1.2 Deriving the relation for k

Next, we use the second Landau equation (momentum-balance condition)

$$\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 = 0\,. \tag{18}$$

Substituting Eq. (14) into Eq. (18), we get

$$\alpha_1 k + \alpha_2(k+p_1) + \alpha_3(k-p_3) = 0$$

$$(\alpha_1+\alpha_2+\alpha_3)k + \alpha_2 p_1 - \alpha_3 p_3 = 0\,. \tag{19}$$

Since the Feynman parameters satisfy the relation

$$\alpha_1 + \alpha_2 + \alpha_3 = 1\,,$$

Eq. (19) reduces to

$$k + \alpha_2 p_1 - \alpha_3 p_3 = 0\,. \tag{20}$$

Hence, solving for k, we find

$$k = \alpha_3 p_3 - \alpha_2 p_1\,. \tag{21}$$

This expression for k will be used in the following steps to substitute into the on-shell conditions and derive the explicit form of the singular surface for the triangle graph.

5.2 Solving for the parameters

Substituting Eq. (21) into Eqs. (16) and (17), and simplifying, we get two linear relations among α_2 and α_3:

$$2p_1^2\alpha_2 - 2(p_1\cdot p_3)\alpha_3 = p_1^2\,, \tag{22}$$

$$2(p_1\cdot p_3)\alpha_2 - 2p_3^2\alpha_3 = -p_3^2\,. \tag{23}$$

Solving these two equations for α_2 and α_3, we obtain:

$$\alpha_2 = \frac{p_3^2 \left[p_3 \cdot (p_1 + p_3)\right]}{2p_1^2 p_3^2 - 2(p_1 \cdot p_3)^2}\,, \tag{24}$$

$$\alpha_3 = \frac{p_1^2 \left[p_1 \cdot (p_1 + p_3)\right]}{2p_1^2 p_3^2 - 2(p_1 \cdot p_3)^2}\,. \tag{25}$$

5.3 Condition for singularity

Substituting Eq. (21) into $k^2 = m^2$, we have

$$\begin{aligned} k^2 &= (\alpha_3 p_3 - \alpha_2 p_1) \cdot (\alpha_3 p_3 - \alpha_2 p_1) \\ &= \alpha_3^2 p_3^2 - 2\alpha_2 \alpha_3 (p_1 \cdot p_3) + \alpha_2^2 p_1^2\,. \end{aligned} \tag{26}$$

Now, using Eq. (26) together with $k^2 = m^2$, we obtain the constraint:

$$\alpha_3^2 p_3^2 - 2\alpha_2 \alpha_3 (p_1 \cdot p_3) + \alpha_2^2 p_1^2 = m^2\,. \tag{27}$$

Substituting the expressions for α_2 and α_3 from Eqs. (24)–(25) into Eq. (27), and simplifying, gives the condition for the singular surface:

$$(p_1^2)^2 (p_2 \cdot p_3)^2 (p_3^2) - 2p_3^2 (p_1 \cdot p_2) p_1^2 (p_2 \cdot p_3)^2 + (p_3^2)^2 (p_1 \cdot p_2)^2 p_1^2 = 4m^2 \left[(p_2 \cdot p_3)^2 - (p_1 \cdot p_3)^2\right]^2. \tag{28}$$

Equation (28) represents the equation of the **singular hypersurface** in the external momentum variables p_i. Points lying on this surface correspond to the possible singularities of the Feynman integral associated with the triangle graph.

6 Application to the Square Graph

Next, we extend our analysis of the Landau equations to the **square (box) graph**. In this case, the Feynman integral contains four internal lines, and we aim to determine the condition for the singular surface in the external momentum space. As before, we assume that all internal and external lines have the same mass m, and that all external momenta are incoming.

6.1 Setup and conventions

By momentum conservation,

$$p_1 + p_2 + p_3 + p_4 = 0\,. \tag{29}$$

We label the internal momenta along the loop as

$$q_1 = k, \qquad q_2 = k + p_1, \qquad q_3 = k + p_1 + p_2, \qquad q_4 = k - p_4\,, \tag{30}$$

where k is the loop momentum.

6.2 Applying the Landau equations

The first Landau equations (on-shell conditions) for the four internal lines are:

$$q_1^2 - m^2 = 0 \quad \Rightarrow \quad k^2 = m^2\,, \tag{31}$$

$$q_2^2 - m^2 = 0 \quad \Rightarrow \quad (k + p_1)^2 = m^2\,, \tag{32}$$

$$q_3^2 - m^2 = 0 \quad \Rightarrow \quad (k + p_1 + p_2)^2 = m^2\,, \tag{33}$$

$$q_4^2 - m^2 = 0 \quad \Rightarrow \quad (k - p_4)^2 = m^2\,. \tag{34}$$

Expanding the terms and simplifying using Eq. (31), we obtain:

$$p_1^2 + 2p_1 \cdot k = 0\,, \tag{35}$$

$$(p_1 + p_2)^2 + 2(p_1 + p_2) \cdot k = 0\,, \tag{36}$$

$$p_4^2 - 2p_4 \cdot k = 0\,. \tag{37}$$

6.3 Using the second Landau equation

The second Landau equation expresses the momentum-balance condition for the loop:

$$\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 + \alpha_4 q_4 = 0\,. \tag{38}$$

Substituting the explicit forms of q_i from Eq. (30), we get

$$\begin{gathered} \alpha_1 k + \alpha_2(k + p_1) + \alpha_3(k + p_1 + p_2) + \alpha_4(k - p_4) = 0 \\ (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)k + \alpha_2 p_1 + \alpha_3(p_1 + p_2) - \alpha_4 p_4 = 0\,. \end{gathered} \tag{39}$$

For the Feynman parameters, we use the condition

$$\sum_{i=1}^{4} \alpha_i = 1\,,$$

which simplifies Eq. (39) to

$$k = \alpha_4 p_4 - \alpha_2 p_1 - \alpha_3 (p_1 + p_2)\,. \tag{40}$$

6.4 Condition for the singular surface

Substituting Eq. (40) into the on-shell conditions (31–34) and simplifying gives a set of three linear equations for α_2, α_3, and α_4. These equations can be written as

$$2p_1^2\alpha_2 + 2[p_1 \cdot (p_1 + p_2)]\alpha_3 - 2(p_1 \cdot p_4)\alpha_4 = p_1^2\,, \tag{41}$$

$$2(p_1 \cdot (p_1 + p_2))\alpha_2 + 2(p_1 + p_2)^2\alpha_3 - 2[(p_1 + p_2) \cdot p_4]\alpha_4 = (p_1 + p_2)^2\,, \tag{42}$$

$$2(p_1 \cdot p_4)\alpha_2 + 2[(p_1 + p_2) \cdot p_4]\alpha_3 - 2p_4^2\alpha_4 = -p_4^2\,. \tag{43}$$

This system can be solved using Cramer's rule:

$$\alpha_2 = \frac{\Delta_{\alpha_2}}{\Delta}, \qquad \alpha_3 = \frac{\Delta_{\alpha_3}}{\Delta}, \qquad \alpha_4 = \frac{\Delta_{\alpha_4}}{\Delta}\,, \tag{44}$$

where Δ and the numerators Δ_{α_i} are determinants formed from the coefficients of Eqs. (41)–(43). Their explicit forms are algebraically lengthy and can be found by straightforward computation.

6.5 Explicit forms of the determinants

The explicit forms of the determinants appearing in Eq. (44) are given by

$$
\begin{aligned}
\Delta = &-8p_1^2(p_1+p_2)^2p_4^2 + 8p_1^2\big[(p_1+p_2)\cdot p_1\big]^2 + 8\big[p_1\cdot(p_1+p_2)\big]p_4^2 \\
&+ 8(p_1\cdot p_4)^2(p_1+p_2)^2 - 16\big[p_1\cdot(p_1+p_2)\big](p_1\cdot p_4)(p_1\cdot p_2)\,,
\end{aligned} \tag{45}
$$

$$
\begin{aligned}
\Delta_{\alpha_2} = &-4p_1^2(p_1+p_2)^2p_4^2 + 4p_1^2\big[(p_1+p_2)\cdot p_4\big]^2 + 4\big[p_1\cdot(p_1+p_2)\big](p_1+p_2)^2p_4^2 \\
&+ 4p_4^2\big[p_1\cdot(p_1+p_2)\big]\big[(p_1+p_2)\cdot p_4\big] - 4(p_1\cdot p_4)(p_1+p_2)^2\big[(p_1+p_2)\cdot p_4\big] \\
&- 4p_4^2(p_1\cdot p_4)(p_1+p_2)^2\,,
\end{aligned} \tag{46}
$$

$$
\begin{aligned}
\Delta_{\alpha_3} = &-4p_1^2(p_1+p_2)^2p_4^2 - 4p_1^2p_4^2\big[(p_1+p_2)\cdot p_4\big] + 4\big[p_1\cdot(p_1+p_2)\big]p_1^2p_4^2 \\
&- 4p_1^2(p_1\cdot p_4)\big[(p_1+p_2)\cdot p_4\big] + 4(p_1\cdot p_4)p_4^2\big[(p_1+p_2)\cdot p_1\big] \\
&+ 4(p_1\cdot p_4)^2(p_1+p_2)^2\,,
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
\Delta_{\alpha_4} = &-4p_1^2(p_1+p_2)^2p_4^2 - 4p_1^2(p_1+p_2)^2\big[(p_1+p_2)\cdot p_4\big] + 4\big[p_1\cdot(p_1+p_2)\big]^2p_4^2 \\
&+ 4\big[p_1\cdot(p_1+p_2)\big](p_1+p_2)^2(p_1\cdot p_2) + 4\big[p_1\cdot(p_1+p_2)\big]p_1^2\big[(p_1+p_2)\cdot p_4\big] \\
&- 4p_1^2(p_1+p_2)^2(p_1\cdot p_2)\,.
\end{aligned} \tag{48}
$$

Finally, substituting Eq. (40) into the on-shell condition $k^2 = m^2$ gives

$$
(\alpha_4 p_4 - \alpha_2 p_1 - \alpha_3(p_1+p_2))^2 = m^2\,. \tag{49}
$$

Equation (49) represents the equation of the **singular hypersurface** for the square (box) graph. The α_i's are determined by the linear system in Eqs. (41)–(43). Points in external momentum space that satisfy this condition correspond to possible singularities of the Feynman integral.

7 Polkinghorne et al. Derivation for Landau Singularities

In this section, we outline the derivation given by Polkinghorne *et al.* for obtaining the general representation of Feynman integrals and studying their analytic structure, which ultimately leads to the Landau conditions for singularities. The approach relies on the **Schwinger parametrisation**, providing an alternative route to transform complicated Feynman integrals involving multiple propagators into solvable Gaussian forms.

7.1 Schwinger Parametrisation

We begin with the identity

$$
\frac{1}{A} = \int_0^\infty d\nu\, e^{-\nu A}, \tag{50}
$$

which holds for $\mathrm{Re}\,A > 0$. Applying this to each propagator in a product of inverses, we get

$$\prod_{i=1}^{N} \frac{1}{A_i} = \left(\prod_{i=1}^{N} \int_0^\infty d\nu_i \right) e^{-\sum_{i=1}^{N} A_i \nu_i}, \tag{51}$$

where ν_i are the Schwinger parameters associated with each propagator.

7.2 Application to Feynman Integrals

For a general Feynman diagram with n propagators and m loops, the integral can be written as

$$f(p_k) = \# \lim_{\epsilon \to 0^+} \int_0^\infty \prod_{i=1}^{n} d\nu_i \int \prod_{j=1}^{m} d^D k_j \ \exp\left[-\sum_{i=1}^{n} \nu_i (q_i^2 - m_i^2 + i\epsilon) \right], \tag{52}$$

where $\#$ denotes an overall proportionality constant. The integrand in Eq. (52) is Gaussian in the loop momenta k_j, which allows for explicit evaluation.

After expanding the quadratic terms in $q_i(k_j, p_k)$ and collecting similar terms, each integral over k_j takes the form

$$\int_{-\infty}^{\infty} e^{-a k_j^2 + b k_j} \, dk_j = \sqrt{\frac{\pi}{a}} \, e^{b^2/(4a)}, \qquad a > 0. \tag{53}$$

For a four-dimensional momentum k_j, the Gaussian integral contributes a factor of $\left(\frac{\pi}{a_j}\right)^2$. For m loop momenta, this gives an overall factor proportional to $\prod_{j=1}^{m} a_j^{-2}$ in the denominator.

7.3 Scaling Transformation

We now introduce the scaling transformation

$$\nu = \sum_{i=1}^{n} \nu_i, \qquad \alpha_i = \frac{\nu_i}{\nu}, \tag{54}$$

so that $\sum_{i=1}^{n} \alpha_i = 1$. The measure transforms as

$$\prod_{i=1}^{n} d\nu_i = \nu^{n-1} d\nu \prod_{i=1}^{n} d\alpha_i \, \delta\left(1 - \sum_{i=1}^{n} \alpha_i \right). \tag{55}$$

Substituting this back into the integral, we get

$$f(p_j) = \# \lim_{\epsilon \to 0^+} \int_0^1 \prod_{i=1}^{n} d\alpha_i \ \phi(\alpha_i) \, \delta\left(1 - \sum_{i=1}^{n} \alpha_i \right) \int_0^\infty d\nu \ \nu^{n-2m-1} \, e^{-\nu \sum_{i=1}^{n} \alpha_i A'_i}, \tag{56}$$

where $\phi(\alpha_i)$ is a smooth function obtained after Gaussian integration, and A_i' are linear combinations of the external scalar products $p_{jk} = p_j \cdot p_k$ with coefficients that depend on α_i.

7.4 Evaluation of the ν-integral

The remaining ν-integration is a Gamma integral:

$$\int_0^\infty \nu^{z-1} e^{-b\nu}\, d\nu = \frac{\Gamma(z)}{b^z}, \tag{57}$$

with $z = n - 2m$. Applying Eq. (57) to Eq. (56) yields

$$f(p_j) = \# \lim_{\epsilon\to 0^+} \Gamma(n-2m) \int_0^1 \prod_{i=1}^n d\alpha_i \, \frac{\delta(1 - \sum_{i=1}^n \alpha_i)\ \phi(\alpha_i)}{\left(\sum_{i=1}^n \alpha_i A_i' + i\epsilon\right)^{n-2m}}. \tag{58}$$

But this is exactly in the form of a Feynman parameterised integrals. Hence, the α_i's are the familiar Feynman parameters. Moreover, the A_i''s are functions of external scalar products p_{jk}. Thus we finally have

$$f(p_{jk}) = \# \lim_{\epsilon\to 0^+} \Gamma(n-2m) \left(\prod_{i=1}^n \int_0^1 d\alpha_i\right) \frac{\delta(1 - \sum_{i=1}^n \alpha_i)\, \phi(\alpha_i)}{[F'(\alpha_i, p_{jk}) + i\epsilon]^{n-2m}} \tag{59}$$

Equation (58) is precisely the Feynman parameter form of the amplitude, showing that after the Schwinger parametrisation, Gaussian integration, and scaling transformation, one obtains an integral over the simplex of α_i with a single denominator raised to a power $n - 2m$. This result corresponds to the general Eq. (3) of Polkinghorne *et al.*, forming the foundation for deriving the Landau singularity conditions.

7.5 Landau Equations in this case

From the parametric form of the Feynman integral in Eq. (58), the singularities arise from the zeros of the denominator, where

$$D = \sum_{i=1}^n \alpha_i (q_i^2 - m_i^2) + i\epsilon = 0.$$

The denominator becomes zero if either $q_i^2 = m_i^2$ or $\alpha_i = 0$. Trapping, or the occurrence of a true singularity, happens when the poles in the complex k_j–plane pinch the contour of integration. This occurs when the derivative of D with respect to each loop momentum k_j vanishes:

$$\frac{\partial}{\partial k_j} \sum_{i=1}^n \alpha_i (q_i^2 - m_i^2) = 0.$$

Since each q_i is linear in the loop momenta, this gives

$$\sum_{i=1}^{n} \alpha_i q_i = 0 \qquad \text{for each loop } j.$$

Thus, the complete set of Landau equations is

$$q_i^2 = m_i^2 \text{ (for all active lines with } \alpha_i \neq 0\text{)}, \qquad \sum_i \alpha_i q_i = 0,$$

which specify the necessary conditions for the possible singularities of the Feynman integral.

8 Outlook

The study of Landau singularities continues to be an active area of research in modern high-energy theory. Recent works have connected the classical ideas of Landau with new tools from geometry and algebra. In particular, computational algebraic geometry is now being used to classify and locate singularities in complicated multi-loop and multi-scale Feynman integrals. Geometric approaches also interpret these singularities as boundaries or faces of Newton or Feynman polytopes (see arXiv:2311.14669 and arXiv:2407.13738).

There is also growing interest in linking Landau singularities with the symbol alphabets that appear in polylogarithmic and bootstrap studies of scattering amplitudes (see arXiv:2104.12776). More recently, algebraic methods using D-ideals have been developed to describe the singular structures of higher-loop integrals in dimensional regularization (see arXiv:2508.04309).

Even with these advances, several open questions remain. One major challenge is identifying which Landau solutions actually correspond to physical thresholds or hidden pinch surfaces, especially in non-planar or massive diagrams. Connecting this geometric and algebraic understanding with practical calculations of amplitudes and their physical interpretation is still an ongoing effort.

Overall, the Landau framework has evolved from a classical tool for locating singularities into a modern bridge between quantum field theory, geometry, and algebra, and it continues to guide new insights into the analytic structure of scattering amplitudes.

9 References

1. J. C. Polkinghorne and G. R. Screaton, *The Analytic Properties of Perturbation Theory I*, *Il Nuovo Cimento*, Vol. XV, No. 2 (1960), pp. 289–307.

2. R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix*, Cambridge University Press (1966).

3. G. Sterman, *An Introduction to Quantum Field Theory*, Cambridge University Press (1993).