ICTP Summer Internship Report
Properties of Scattering Amplitudes

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1 Introduction

Quantum mechanics deals with systems where the number of particles is fixed. In contrast, high-energy physics involves processes where the number of particles is not conserved, such as the production and annihilation of particles in high-energy collisions. Hence, to describe these phenomena, a more general framework is required - quantum field theory. In this framework, any kind of interaction between particles is referred to as a scattering process. These processes are not deterministic and measurable quantities are derived from probability amplitudes. These amplitudes are computed using the S-matrix, which relates initial and final quantum states. Since most quantum field theories cannot be solved exactly, scattering amplitudes are calculated perturbatively: the interaction is expanded in powers of the coupling constant. Each term in the expansion is represented by a Feynman diagram. By applying Feynman rules, each Feynman diagram yields a mathematical expression for the scattering amplitude. However, diagrams often contain internal particles whose momenta are not fixed by external particles. To account for all possible values of these internal momenta, we must integrate over them. These integrals are called Feynman integrals. A generic Feynman diagram involves many such loop momenta and hence is a multiple integral over the loop momenta variable. Once the loop momenta are integrated, the integral becomes a function of the external momenta. However, these integrals are often divergent or ill-defined if we try to evaluate them directly at our points of interest for physics purposes. Therefore, we study the Feynman integrals as functions of (several) complex external momenta variables and analytically continue these functions/integrals to a bigger domain (if possible), which usually contains points of our interest. Analytic continuation means extending the integral to a region where it is mathematically well-defined, and then continuing the result back to the physical region. In order to do so, it is essential to know about the complex singularities of the function. Landau equations give us the conditions to locate these singularities. In this internship, we studied a classic paper [5] which gives an analytic proof of the Landau singularities and extends these ideas for complex singularities.

2 Feynamnn Integrals

The general components of a Feynman diagram are as follows:

- **Lines**: Each line represents a particle. External lines represent incoming or outgoing particles.
- **Vertices**: Vertices are points where lines meet. Each vertex represents an interaction between the particles represented by lines that meet at that vertex. Momentum and energy are conserved at each vertex.

The Feynman diagrams corresponding to a given scattering process can be of many kinds: tree-level, 1-loop, 2-loop etc. The diagrams can also be planar or non-planar.

To compute the scattering amplitude, in theory, all the Feynman diagrams possible for that interaction have to be taken into account. The Feynman rules tell us how to associate a complex number to each Feynman diagram. These rules are as follows:

- Add a momentum k_j to any internal line in a loop. These are the loop momenta. By conveniently selecting a set of closed loops that specify the structure of the graph, a set of independent loop momenta associated with the graph can be generated.[1]
- Applying conservation of energy-momentum in each vertex, express the momenta q_i of all the virtual particles in the loop using the loop momenta k_j and the external momenta p_k. Then q_i will be a linear function of k_j and p_k.
- To each internal line, a propagator is associated: $\frac{i}{q_i^2 - m_i^2 + i\epsilon}$

  Here q_i is the momentum of the internal line and m_i is the mass.
- Then the resulting expression is integrated over the loop momenta: $\int \frac{d^4 k_j}{(2\pi)^4} \frac{i}{q_i^2 - m_i^2 + i\epsilon}$

The above rules are applied for each loop momenta and the resulting final integral is in fact a multiple integral in the loop momenta. However, the denominator of the integrand involves a product of many propagator factors. To simplify such integrals, a mathematical trick known as Feynman parametrisation is used. It allows one to combine multiple propagator factors in the denominator into a single denominator raised to a power.

[1] These momenta are actually energy-momentum, $p = (p^0, \vec{p})$. Here, the first component (p^0) denotes energy and the last three components denote the spatial components of momentum ($\vec{p}$).

The basic idea of the parametrisation can be understood by employing it in the simple case of a denominator with two propagators. It is based on the following basic identity:

$$\frac{1}{AB} = \int_0^1 dx \, \frac{1}{\left[xA + (1-x)B\right]^2} \tag{1}$$

For the case with n propagators, ignoring the constant factors, the Feynman parametrisation yields:

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 d\alpha_1 \cdots d\alpha_n \, \delta\Big(1 - \sum_{i=1}^{n} \alpha_i\Big) \, \frac{1}{\Big(\sum_{i=1}^{n} \alpha_i A_i\Big)^n}. \tag{2}$$

Here the variables $\alpha_1, \ldots, \alpha_N$ are Feynman parameters.

With the help of Feynman parametrisation, the Feynman integrals have the following general form:

$$\lim_{\epsilon \to 0^+} \int_0^1 d\alpha_1 \ldots d\alpha_n \int d^4k_1 \ldots d^4k_m \frac{\delta\Big(\alpha_1 + \cdots + \alpha_n - 1\Big)}{[F(\alpha_i, k_j, p_k) + i\epsilon]^n} \tag{3}$$

where

$$F(\alpha_i, k_j, p_k) = \sum_{i=1}^{n} \alpha_i (q_i^2 - m_i^2) \tag{4}$$

If the integrations over the loop momenta k_j are performed, then we have an integral of the following form[2]

$$\lim_{\epsilon \to 0^+} \int_0^1 d\alpha_1 \ldots d\alpha_N \frac{\phi(\alpha_i)\delta\Big(\alpha_1 + \cdots + \alpha_N - 1\Big)}{[F'(\alpha_i, p_j k) + i\epsilon]^{n-2m}} \tag{5}$$

We have omitted numerical multiplicative factors in the above expression. The function F' is a function of the external scalar products $p_{jk} = p_j \cdot p_k$ and is obtained from the function F by eliminating the k_j's. Our goal is to investigate the analytic properties of the above expression when the external scalar products are regarded as complex variables.

[2]The explicit calculation for obtaining (5) from (3) is shown in Appendix B.

3 Singularities of Integral representations

As mentioned in the introduction section, to study the analytic properties of Feynman integrals, it is important to know about the position of the singularities of the function. Since the functional representation in this case is in terms of an integral, we study a lemma outlined in [5] which discusses the circumstances in which a function defined by an integral may have a singularity. We begin with classifying the types of singularities that may arise and developing the main ideas behind the lemma, starting with the simple case of simple integrals of one variable and then extending the argument for several variables and multiple integrals. [3]

3.1 Singularities of Simple Integrals

Let $g(z, w)$ be an analytic function of two complex variables and let C be some finite contour in the complex w-plane. We define a function $f(z)$ by

$$f(z) = \int_C g(z, w) dw \tag{6}$$

We suppose that the locations of the singularities of the integrand g are known and their positions in w-plane are

$$w = w_r(z) \quad (r = 1, 2, \dots) \tag{7}$$

We further suppose that for z in a neighbourhood of some point z_0 there is a neighbourhood of the contour C in the w-plane free from the singularities w_r. Then the integral in (6) becomes well-defined and $f(z)$ is anaytic at z_0. We wish to analytically continue $f(z)$ away from z_0 and investigate how a singularity of $f(z)$ can arise in such a case and how it will affect our scheme of analytic continuation.

As z is moved away from z_0, the singularities w_r (which are functions of z) will move about in the w-plane. But the contour integral (6) can still be evaluated as long as none of the singularities w_r reach C and hence $f(z)$ will remain anaytic. However, when a singularity does reach the contour C, the integral (6) becomes undefined. But $f(z)$ can still be further analytically continued in such a case by employing a scheme that we describe below. This scheme is based on the Cauchy-Goursat theorem in complex analysis. The

[3] The exposition in this section follows Chapter 2 from [1]

theorem states that if the region enclosed by a closed contour doesn't contain any pole or singularity, then the contour integral is zero. Now suppose that C has endpoints A and B. Let us suppose at first that the endpoints are fixed. Then, if C' is another contour with the same end-points and if both C' and the region between C and C' don't contain any singularity, then the Cauchy-Goursat theorem implies that the contour integrals in (6) along contours C and C' are the same.

$$f(z) = \int_{C'} g(z, w) dw \tag{8}$$

Hence, as we analytically continue $f(z)$ from z_0 to some point z along some

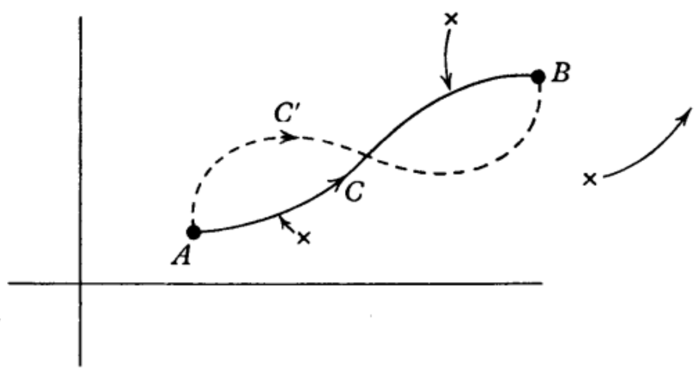

Figure 1: Deformation of the contour in the complex w-plane.

path, successive points along that path define successive contours in this way, with the property that the contours define functions that are analytic at the point in question. The conotours are obtained by continuously deforming the contour C away from the direction of the advancing singularities. These associated functions provide a continuation of $f(z)$ from z_0 to any point z.

This is illustrated in Figure 1. The crosses here represent positions of the singularities at the initial point z_0 and the arrows indicate how the singularities might have moved during the variation of z from z_0 along the path.

The above procedure may fail for one of three reasons[4]:

i. *End-point singularities*: If one of the singularities w_r reaches one of the end-points A, B of the contour C, then no variation of the contour will

[4]Simple examples of these possibilities are given in Appendix A

allow us to avoid the singularities. These singularities are known as end-point singularities.

ii. *Pinching singularities*: If two or more singularities approach the contour from opposite sides and coincide, then the contour C becomes pinched between them and no deformation of C can avoid them (See Figure 2). Alternatively, one of the singularities may be fixed and the pinching may arise as a result of the other approaching it, with C between.

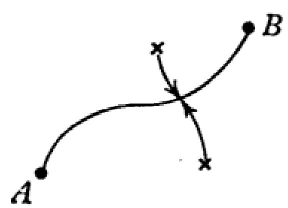

Figure 2: Pinching singularities

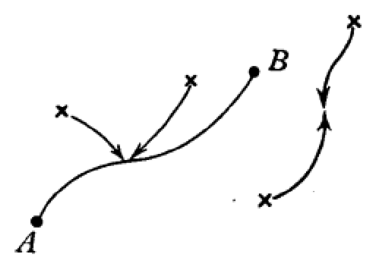

Figure 3: Harmless pinching

It is important to emphasise that pinching occurs only if the singularities approach each other from the opposite sides of the contour. If they approach from the same side or if they come together nowhere near the contour, as in Figure 3, they are harmless. Finding whether the contour is trapped is actually the difficult part of the analysis.

iii. *Infinite deformations*: If the contour C is being deformed to avoid a singularity and that singularity moves off to infinity, dragging C with it, the integral may diverge because C is no longer finite. By making a transformation of the integration variable, such as $w = \frac{1}{\zeta}$, the point at infinity can be brought into the finite part of the complex plane. Hence, these singularities can be reduced to a special case of (ii) and therefore, are not separately classified in what follows.

In the above discussion, we supposed that the end-points of the contour were fixed. However, no new feature is added in principle if one or both of the end-points are allowed to be a function of z. Pinching may occur as before, while an end-point singularity can occur at $z = z_1$, if for some r,

$$\text{or if} \qquad \left.\begin{aligned} w_r(z_1) &= A(z_1), \\ w_r(z_1) &= B(z_1). \end{aligned}\right\} \tag{9}$$

Again, the contour C may be closed and have no end-points at all. Only pinch singularities are encountered in this case. Finally, since by a transformation of variables it can be seen that a point at infinity is no different from any other point, we may drop the restriction that C be finite.[5]

3.2 More than One External Variable

The above discussion generalises in a straightforward manner to a function of more than one complex variable defined by a simple integral. As an example, let's consider

$$f(z, z') = \int_C g(z, z', w)dw \tag{10}$$

Here, the positions of the singularity w_r of the integrand are now generally functions of two variables z and z'. However, the conditions for singularity are just as before. Thus, end-point singularities are given by

$$w_r(z, z') = A \tag{11}$$

for some r and w_1 and w_2 can produce a pinch singularity when

$$w_1(z, z') = w_2(z, z') \tag{12}$$

Each of (11) and (12) defines a two-dimensional surface in the four-dimensional space of the complex variables z, z'.

3.3 Singularities of Multiple Integrals

The previous discussion may be generalised to multiple integrals of the form

$$f(z) = \int_H \Pi \, dw_i g(z, w_i). \tag{13}$$

Here, the contour of integration C of the simple integral case has become a 'hypercontour' H in the multi-dimensional complex w_i-space.

[5]The figures in this section are taken from [1]

The singularities of the integrand $g(z, w_i)$ are imagined as being given by various equations

$$S_r(z, w_i) = 0 \quad (r = 1, 2, \dots) \tag{14}$$

For any value of z a given S_r will be a $(2n-2)$-dimensional surface in the $2n$-dimensional complex w_i-space. When z is varied away from its natural domain, these S_r will move around till one of them comes to intersect H. To avoid this, H must be distorted away from its original real location, much as the simple contour C was distorted for the case of the single, integration variable. When the possibility of this distortion ceases, so that H becomes trapped, a singularity of $f(z)$ is encountered. It is intuitive to suppose that f will be singular only if each integration in (13) has either an end-point singularity or a pinching singularity. This turns out to be true but we will omit the argument that leads to it.

4 Landau equations

In this section, we derive the Landau equations, which provide necessary conditions for the locations of singularities of Feynman integrals. A full, rigorous proof would require tools from topology, but here we will be content with plausibility-based arguments. It is also important to emphasize that the Landau equations give only necessary, not sufficient, conditions; therefore, they do not guarantee the existence of a singularity, but only indicate where one may potentially occur.

The singularities of the integral (3) must arise from the zeros of the denominator. Now the denominator will be zero if for each i we have

$$\text{either} \quad q_i^2 = m_i^2, \quad \text{or} \quad \alpha_i = 0 \tag{15}$$

Now the denominator in (3) is quadratic in the loop momenta k_j. For fixed values of other variables, the denominator is analytic in the k_j-plane, except for the two poles which are found by solving the quadratic equation

$$D = \sum_{i=1}^{n} \alpha_i (q_i^2 - m_i^2) + i\epsilon = 0 \tag{16}$$

Trapping will occur only if these two solutions are equal. This will happen when the derivative of D with respect to k_j vanishes at $D = 0$:

$$\frac{\partial}{\partial k_j} \sum_i \alpha_i (q_i^2 - m_i^2) = 0 \tag{17}$$

for each loop momentum integration variable k_j. Since each q_i is a linear combination of the k_j's, this becomes

$$\sum_j \alpha_i q_i = 0 \quad \text{for each } j, \tag{18}$$

where $\sum_j$ denotes summation around the loop around which k_j runs. The equations (15) and (18) are the Landau equations.[6]

5 Application to Triangle and Square Graph

As an application of the Landau equations, we consider the triangle and the square graphs when all external and internal masses are the same and determine the equation of the hypersurfaces in the p_k variables that may contain the singularities. We do not consider any $\alpha_i = 0$. We also do not consider the on-shell condition on the external momenta. Furthermore, we use the convention that all external momenta are incoming.[7]

[6]The material in this section is based on the discussion in Chapter 13 of [7].

[7]The diagrams in Figure 4 and Figure 5 are due to the courtesy of my friend Jathiya Fin-Nur Bintay Huq.

5.1 Triangle Graph

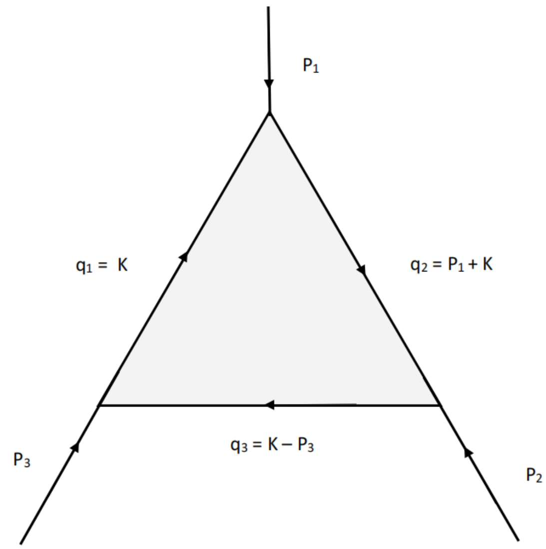

Figure 4: Triangle diagram.

By our convention,

$$p_1 + p_2 + p_3 = 0 \tag{19}$$

Let m be the mass common to all internal and external lines. Then, applying the first Landau equations to each of the internal lines yields

$$k^2 = m^2 \tag{20}$$

$$(k + p_1)^2 = m^2 \Rightarrow \quad p_1^2 + 2p_1 \cdot k = 0 \tag{21}$$

$$(k - p_3)^2 = m^2 \Rightarrow \quad p_3^2 - 2p_3 \cdot k = 0 \tag{22}$$

where we used (20) to simplify (21) and (22).

Applying the second Landau equation yields the following:

$$\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 = 0 \tag{23}$$

Using the expressions for the loop momenta q's from the diagram and using the fact that $\sum_{i=1}^{3} \alpha_i = 1$, we have

$$k = \alpha_3 p_3 - \alpha_2 p_1 \tag{24}$$

Substituting this into (21) and (22) and simplifying,

$$2p_1^2\alpha_2 - 2(p_1 \cdot p_3)\alpha_3 = p_1^2 \tag{25}$$

$$2(p_1 \cdot p_3)\alpha_2 - 2p_3^2\alpha_3 = -p_3^2 \tag{26}$$

Solving (25) and (26) for α_2 and α_3,

$$\alpha_2 = \frac{p_3^2(p_3 \cdot (p_1 + p_3))}{2p_1^2 p_3^2 - 2(p_1 \cdot p_3)^2} \tag{27}$$

$$\alpha_3 = \frac{p_1^2(p_1 \cdot (p_1 + p_3))}{2p_1^2 p_3^2 - 2(p_1 \cdot p_3)^2} \tag{28}$$

Substituting these back into (24),

$$\begin{aligned} k^2 &= (\alpha_3 p_3 - \alpha_2 p_2) \cdot (\alpha_3 p_3 - \alpha_2 p_2) \\ &= \alpha_3^2 p_3^2 - 2\alpha_2\alpha_3(p_1 \cdot p_3) + \alpha_2^2 p_1^2 \end{aligned} \tag{29}$$

Finally using (20) and (29),

$$\alpha_3^2 p_3^2 - 2\alpha_2\alpha_3(p_1 \cdot p_3) + \alpha_2^2 p_1^2 = m^2 \tag{30}$$

$$\Rightarrow (p_1^2)^2(p_2 \cdot p_3)^2(p_3^2) - 2p_3^2(p_1 \cdot p_2)p_1^2(p_2 \cdot p_3)^2 + (p_3^2)^2(p_1 \cdot p_2)^2 p_1^2 = 4m^2\left[(p_2 \cdot p_3)^2 - (p_1 \cdot p_3)^2\right]^2 \tag{31}$$

Equation (31) is the equation of the hypersurface that may contain singularities of the triangle graph.

5.2 Square Graph

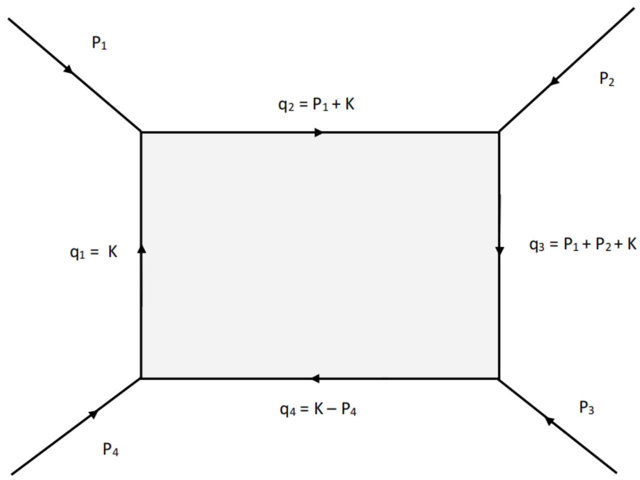

Figure 5: Square diagram.

By our convention,

$$p_1 + p_2 + p_3 + p_4 = 0 \tag{32}$$

Let m be the mass common to all internal and external lines. Then, applying the first Landau equations to each of the internal lines yields

$$k^2 = m^2 \tag{33}$$

$$(k + p_1)^2 = m^2 \Rightarrow \quad p_1^2 + 2p_1 \cdot k = 0 \tag{34}$$

$$(k + p_1 + p_2)^2 = m^2 \Rightarrow \quad (p_1 + p_2)^2 + 2(p_1 + p_2) \cdot k = 0 \tag{35}$$

$$(k - p_4)^2 = m^2 \Rightarrow \quad p_4^2 - 2p_4 \cdot k = 0 \tag{36}$$

where we used (33) to simplify (34), (35) and (36). Applying the second Landau equation yields the following:

$$\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 = 1 \tag{37}$$

Using the expressions for the loop momenta q's from the diagram and using the fact that $\sum_{i=1}^{4} \alpha_i = 0$, we have

$$k = \alpha_4 p_4 - \alpha_2 p_1 - \alpha_3 (p_1 + p_2) \tag{38}$$

Substituting this into (34), (35) and (36) and simplifying,

$$2p_1^2\alpha_2 + 2(p_1 \cdot (p_1 + p_2))\alpha_3 - 2(p_1 \cdot p_4)\alpha_4 = p_1^2 \tag{39}$$

$$2(p_1 \cdot (p_1 + p_2))\alpha_2 + 2(p_1 + p_2)^2\alpha_3 - 2((p_4 \cdot (p_1 + p_2))\alpha_4 = p_1 + p_2 \tag{40}$$

$$2p_1 \cdot p_4\alpha_2 + 2((p_1 + p_2) \cdot p_4)\alpha_3 - 2p_4^2\alpha_4 = -p_4^2 \tag{41}$$

These equations constitute a linear system of equations in the variables α_2, α_3 and α_4. Solving them with the help of Cramer's rule yields,

$$\begin{aligned} \alpha_2 &= \frac{\Delta_{\alpha_2}}{\Delta} \\ \alpha_3 &= \frac{\Delta_{\alpha_3}}{\Delta} \\ \alpha_4 &= \frac{\Delta_{\alpha_4}}{\Delta} \end{aligned} \tag{42}$$

where

$$\begin{aligned}
\Delta =& -8\,p_1^2(p_1+p_2)^2p_4^2 + 8\,p_1^2\left((p_1+p_2)\cdot p_1\right)^2 + 8\,(p_1\cdot(p_1+p_2))\,p_4^2 \\
&+ 8\,(p_1\cdot p_4)^2(p_1+p_2)^2 - 16\,(p_1\cdot(p_1+p_2))(p_1\cdot p_4)(p_1\cdot p_2) \\
\Delta_{\alpha_2} =& -4\,p_1^2(p_1+p_2)^2p_4^2 + 4\,p_1^2\left((p_1+p_2)\cdot p_4\right)^2 \\
&+ 4\,(p_1\cdot(p_1+p_2))\,(p_1+p_2)^2p_4^2 + 4\,p_4^2\,(p_1\cdot(p_1+p_2))\,((p_1+p_2)\cdot p_4) \\
&- 4\,(p_1\cdot p_4)\,(p_1+p_2)^2((p_1+p_2)\cdot p_4) - 4\,p_4^2\,(p_1\cdot p_4)\,(p_1+p_2)^2 \\
\Delta_{\alpha_3} =& -4\,p_1^2(p_1+p_2)^2p_4^2 - 4\,p_1^2p_4^2\,((p_1+p_2)\cdot p_4) \\
&+ 4\,(p_1\cdot(p_1+p_2))\,p_1^2p_4^2 - 4\,p_1^2\,(p_1\cdot p_4)\,((p_1+p_2)\cdot p_4) \\
&+ 4\,(p_1\cdot p_4)\,(p_4)^2((p_1+p_2)\cdot p_1) + 4\,\left(p_1\cdot p_4\right)^2(p_1+p_2)^2 \\
\Delta_{\alpha_4} =& -4\,p_1^2(p_1+p_2)^2p_4^2 - 4\,p_1^2(p_1+p_2)^2\,((p_1+p_2)\cdot p_4) \\
&+ 4(p_1\cdot(p_1+p_2))^2p_4^2 + 4\,(p_1\cdot(p_1+p_2))\,(p_1+p_2)^2(p_1\cdot p_2) \\
&+ 4\,(p_1\cdot(p_1+p_2))\,p_1^2((p_1+p_2)\cdot p_4) - 4\,p_1^2(p_1+p_2)^2(p_1\cdot p_2)
\end{aligned} \tag{43}$$

Now using (33) and (38), we have

$$(\alpha_4 p_4 - \alpha_2 p_1 - \alpha_3(p_1 + p_2))^2 = m^2 \tag{44}$$

$$\begin{aligned} \alpha_4^2p_4^2 + \alpha_2^2p_1^2 + \alpha_3^2(p_1+p_2)^2 &- 2\alpha_2\alpha_4(p_1\cdot p_4) - 2\alpha_3\alpha_4(p_4\cdot(p_1+p_2)) \\ &+ 2\alpha_2\alpha_3(p_1\cdot(p_1+p_2)) = m^2 \end{aligned} \tag{45}$$

Equation (45) represents the equation of the hypersurface that may contain singularities of the triangle graph, where the α's are given by (42).

6 Outlook

While the paper [5] we studied dates back to the 1960s, the recent revival of interest in scattering amplitudes and the analytic properties of Feynman integrals has made the study of Landau singularities an important theme in contemporary research. Possible extensions of the present work can be pursued in several directions.

One avenue appears in the study of crossing symmetry, which asserts that a scattering amplitude for one process can be analytically continued to describe another process where incoming particles are replaced by outgoing antiparticles (and vice versa). The paper [4] by Sebastian Mizera investigates how Landau singularities control the analytic continuation between different crossing channels via contour deformations.

A distinct direction is explored in [3], where Gardi et al. study Landau singularities using parametric representations of Feynman integrals. Their analysis leads to a geometric interpretation of singularities as faces of Newton polytopes, suggesting that geometric methods may reveal new structural insights. A closely related line of work appears in [2], where the authors reformulate the analysis of singularities using computational algebraic geometry, providing algorithms that can be implemented to aid perturbative computations in the Standard Model.

These developments illustrate that, despite the contents of [5] being over half a century ago, the ideas of Landau singularities remain deeply relevant, and modern mathematical tools continue to shed new light on the analytic structure of Feynman integrals.

To summarize, we have reviewed the derivation of the Landau equations and examined explicit examples of their solutions in one-loop Feynman diagrams. With the growing interest in scattering amplitude methods and reformulations of Feynman integrals, the study of Landau singularities continues to be a central part of understanding quantum field theory. Further progress in this direction is likely to come from the interplay between physics, geometry, and computational methods, and it will be interesting to see how the subject evolves in the coming years.

A Simple Examples of Different Types of Singularities

i.

$$f(z) = \int_a^b \frac{dw}{w-z} = \log\left(\frac{b-z}{a-z}\right) \tag{46}$$

Here the integrand $(w-z)^{-1}$ is singular at the end-points a, b if $z = a, b$; corresponding to the singularity of the logarithm at these points. This is an example of end-point singularity.

ii.

$$f(z) = \int_0^1 \frac{dw}{(w-z)(w-a)} = \frac{1}{z-a} \log\left\{\frac{a(1-z)}{(1-a)z}\right\} \quad (a > 1) \tag{47}$$

Here the logarithmic singularities $z = 1, 0$ are end-point singularities, while the singularity $z = a$ arises from a pinch between the singularities $w_1 = z$ and $w_2 = a$ of the integrand.

iii.

$$f(z) = \int_2^3 \frac{dw}{zw+1} = \frac{1}{z} \log\left(\frac{3z+1}{2z+1}\right) \tag{48}$$

This function has three singularities: $z = -\frac{1}{2}, -\frac{1}{3}, 0$. The singularities at $z = -\frac{1}{2}, -\frac{1}{3}$ are end-point singularities while the singularity at $z = 0$ is an infinite singularity. To see this, we make the transformation $w = \frac{1}{\zeta}$:

$$f(z) = \int_{\frac{1}{3}}^{\frac{1}{2}} \frac{d\zeta}{\zeta(\zeta+z)} \tag{49}$$

The singularity at $z = 0$ now arises from a pinch at $\zeta = 0$.

B Some Nontrivial Calculations

Here, we explicitly show how to obtain (5) from (3)[8]. Our calculation is based on Schwinger parametrisation, which is an alternative technique to

[8] Some steps of this calculation were guided by [6] and a short note titled "Analytic Feynman Diagrams" by Ratul Mahanta and Tanmoy Sengupta.

transform complicated Feynman integrals involving a product of propagators in the denominator into solvable Gaussian integrals, along with Feynman parametrisation. We begin by noting that

$$\frac{1}{A} = \int_0^\infty d\nu\, e^{-A\nu} \tag{50}$$

when $\operatorname{Re} A > 0$. Applying this to each of the terms in a product of inverses,

$$\prod_{i=1}^{n} \frac{1}{A_i} = \left(\prod_{i=1}^{n} \int_0^\infty d\nu_i \right) e^{-\sum_{i=1}^{n} A_i \nu_i} \tag{51}$$

where the ν_i are Schwinger parameters.

When we apply this trick to our Feynman integrals, we have

$$f(p_k) = \# \lim_{\epsilon \to 0^+} \int_0^\infty \prod_{i=1}^{n} d\nu_i \int \prod_{j=1}^{m} d^4 k_j e^{-\sum_{i=1}^{n}(q_i^2 - m_i^2 + i\epsilon)\nu_i} \tag{52}$$

$$= \# \lim_{\epsilon \to 0^+} \int_0^\infty \prod_{i=1}^{n} d\nu_i e^{\sum_{i=1}^{n}(m_i^2 - i\epsilon)\nu_i} \int \prod_{j=1}^{m} d^4 k_j e^{-\sum_{i=1}^{n} \nu_i q_i^2} \tag{53}$$

Where # denotes some constant. The integrand in the loop momenta integrals is in the form of a Gaussian integral with the k_j's being the variables of integration. Here, each q_i is a linear function of the loop momenta k_j and external momenta p_k. If each of these expressions for q_i is substituted and the squares are expanded, then we can collect the k_j's separately and write the exponent as $e^{-M_1(k_1)} \cdots e^{-M_m(k_m)} e^{-Q(p_{jk})}$. Here the functions M's are of the form $M_i(k_i) = a_i k_i^2 + b_i k_i$, where a_i is some linear combination of the ν_i's and b_i represents some linear combination of the p_k's. Hence, each of the exponential factors becomes a Gaussian integral of the following form:

$$\int_{-\infty}^{\infty} e^{-ax^2 + bx}\, dx = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a}}, \quad a > 0 \tag{54}$$

If we evaluate these Gaussian integrals, and combine the $e^{\frac{b_i^2}{4a_i}}$'s together with $e^{Q(p_{jk})}$, then we get an exponential function of the form $e^{Q'(p_{jk})}$ where in some cases we may have $j = k$. Here Q' will be a linear function of the external scalar products $p_{jk} = p_j \cdot p_k$ with the coefficients being some linear combination of ν_i.

Now let $\nu = \sum \nu_i$ and $\alpha_i = \nu_i/\nu$. Then each of the a_i will become a product of ν and some linear combination of the α_i's. Furthermore, the Gaussian integral in (54) is for the case of one dimension. Since the loop momenta k_j are vectors in four dimensions, evaluation of the integration for a single loop momenta variable will give the factor $\sqrt{\frac{\pi}{a_i}}$ four times. There are a total of m loop momenta. Therefore, we will get the factor ν in the denominator with a total power of $m * 4 * \frac{1}{2} = 2m$. Moreover, we can collect the contributions from the α_i's into a single function $\phi(\alpha_i)$. Furthermore, we can combine all the exponentials into one single exponential and separate the parts containing ν_i. Then, with the above transformation, we can write this single exponential in the form $e^{-\nu \sum_{i=1}^{n} \alpha_i A'_i}$.

Now with the above scaling transformation, we have

$$\prod_{i=1}^{n} d\nu_i = \nu^{n-1} d\nu \prod_{i=1}^{n} d\alpha_i \, \delta\left(1 - \sum_{i=1}^{n} \alpha_i\right) \tag{55}$$

so

$$f(p_j) = \# \lim_{\epsilon \to 0^+} \left(\prod_{i=1}^{n} \int_0^1 d\alpha_i \right) \phi(\alpha_i) \delta\left(1 - \sum_{i=1}^{n} \alpha_i\right) \int_0^\infty \nu^{n-2m-1} d\nu \, e^{-\nu \sum \alpha_i A'_i} \tag{56}$$

But

$$\int_0^\infty t^{z-1} e^{-bt} dt = \frac{1}{b^z} \Gamma(z) \tag{57}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt = (z-1)! \tag{58}$$

is the Gamma function. So, altogether,

$$f(p_j) = \# \lim_{\epsilon \to 0^+} \Gamma(n-2m) \left(\prod_{i=1}^{n} \int_0^1 d\alpha_i \right) \frac{\phi(\alpha_i) \delta(1 - \sum_{i=1}^{n} \alpha_i)}{(\sum \alpha_i A'_i)^{n-2m}} \tag{59}$$

But this is exactly in the form of a Feynman parameterised integral. Hence, the α_i's are the familiar Feynman parameters. Moreover, the A'_i's are functions of external scalar products p_{jk} only. Thus, we finally have

$$f(p_{jk}) = \# \lim_{\epsilon \to 0^+} \Gamma(n-2m) \left(\prod_{i=1}^{n} \int_0^1 d\alpha_i \right) \frac{\phi(\alpha_i) \delta(1 - \sum_{i=1}^{n} \alpha_i)}{[F'(\alpha_i, p_{jk}) + i\epsilon]^{n-2m}} \tag{60}$$

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Approval

The internship report titled "**Properties of Scattering Amplitudes**" submitted by **Md. Merajul Hasan Fardin**, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfilment of the requirements of the internship program.

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