Report on PWF Internship Project: Quintessence

Mentee: **M. Fahim Hoque**[a]
Mentor: **Ahmed Rakin Kamal**[b]

[a]*BRAC University, Dhaka 1212, Bangladesh*

[b]*Department of Theoretical Physics and Astrophysics, Faculty of Science, Masaryk University, 611 37 Brno, Czech Republic.*

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1 Overture

The theoretical and phenomenological facets of Quintessence are reviewed in this report. An isotropic, accelerating universe described by a static Cosmological Constant (Λ), a vacuum energy solution, was formerly thought to be the cause of the observed cosmic acceleration. However, this component has serious problems with coincidence and theoretical fine-tuning. It may be possible to use the dark energy component as a good explanation for the cosmic evolution if it is a time-dependent entity. The way that Quintessence and the ΛCDM model both explain the late-time universe explains their relationship. Because the expansion is driven by the kinetic and potential energy of this field, the essence of the dark energy epoch may be captured by generalizing the static Cosmological Constant to a dynamical, rolling scalar field.

The Cosmological Constant is presented and familiarized with in the first section. The rest is expanded upon from the first coverage of the mathematical tools :equation of state parameter, w, stress-energy tensor of the vacuum. The issues in the ΛCDM model are replicated with these prerequisites, particularly the "Old" Cosmological Constant issue involving the 120-order-of-magnitude difference between theoretical prediction and observation.
The concept of a dynamical scalar field is brought about due to the use of rolling scalar field used for early universe inflation. A negative equation of state parameter, $w = -1$, has been used as an alternative for the cosmological constant in the 1980s. The first form of quintessence, the thawing field: a tilted, linear potential in contrast to the flat potential, was proposed by Linde in 1987, that causes the field to roll when the expansion rate of the universe had decreased sufficiently. The action of such a scalar field was introduced as an alternative to the standard ΛCDM model and it depicts the behavior of a canonical field minimally connected to gravity. In order to account for the universe's time-dependent expansion, we let the equation of state parameter change over time instead of staying constant at $w = -1$. We presented the Lagrangian dynamics of this scalar field $\phi(t)$ to determine if it is a rolling scalar field that would enable the time-dependent evolution to generate distinct expansion histories. The resulting solutions are anticipated to give rise to new model classifications (such as Thawing and Freezing) that go beyond the conventional static dark energy model and are essentially considered as potential answers to the coincidence and fine-tuning issues.

2 The Cosmological Constant: A brief overview

2.0.1 What is the Cosmological Constant?

The C.C is the constant added to the Einstein equations to implicate that the universe is indeed static. However, 1.) De Sitter's alternate static universe model without matter which lead to the idea that the expanding universe would not need the C.C, as discovered by Weyl and Eddington, led to Einstein to do 'away with the cosmological term!'.

2.1 Equivalence of the C.C with vacuum energy

We know, the the classical action of the gravitational field, along with the matter action is given by [1]:

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g}(R - 2\Lambda_B) + S_{matter}[g_{\mu\nu}, \Psi] \tag{2.1}$$

where $\kappa \equiv 8\pi G/c^4 \equiv 8\pi/m_{\rm Pl}^2 \equiv 1/M_{\rm Pl}^2$, $m_{\rm Pl}$ and $M_{\rm Pl}$ being the Planck mass and the reduced Planck mass respectively. λ_B is the bare cosmological constant and has the dimension of the inverse of square length.

Ignoring the generic matter field Ψ for now, if we vary the total action w.r.t the metric tensor, we yield the Einstein e.o.m:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda_B g_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.2}$$

where the stress-energy tensor is defined by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\rm matter}}{\delta g^{\mu\nu}} \tag{2.3}$$

Now, the stress-energy tensor of a field placed in the vacuum state is given by:

$$\langle 0|T_{\mu\nu}|0\rangle = -\rho_{vac} g_{\mu\nu} \tag{2.4}$$

It can be quite easily proven that $T_{\mu\nu}^{(vac)} = -\rho_{\rm vac} g_{\mu\nu}$ is indeed true, by analyzing the minimum energy state of a perfect fluid, in which case the the stress energy tensor boils down to above.

Now, taking this and QFT into account, i.e., considering the gravitation of the vacuum fluctuations, the Einstein equations become:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda_B g_{\mu\nu} = \kappa T_{\mu\nu}^{\rm matter} + \kappa \langle T_{\mu\nu} \rangle \tag{2.5}$$

where on the r.h.s. the first contribution comes from ordinary matter while the second one represents the contribution originating from the vacuum. So, the Einstein equations without the C.C, i,e., in a static universe are:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} \left(T_{\mu\nu}^{\rm matter} - \rho_{vac} g_{\mu\nu} \right).$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \left(\frac{8\pi G}{c^4} \rho_{\rm vac} \right) g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}^{\rm matter}. \tag{2.6}$$

Comparing equations (2) and (6), we get:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}^{\rm matter}.$$

and finally,

$$\rho_{vac} = \frac{\Lambda c^4}{8\pi G}. \tag{2.7}$$

This helps reinforce the fact that: $\lambda_{eff} = \lambda_B + \kappa\rho_{vac}$, thus allowing us to conclude the fact that the effective cosmological constant is the sum of the bare cosmological constant and the vacuum fluctuation contributing term .

3 The Problem With the Cosmological Constant

Our present calculated value of the vacuum energy is more than 50 orders of magnitude larger than it should in terms of experimental data. Till now, no concrete theory exists to explain this discrepancy. In nature, scales are independent of each other, i.e., large distance scales are not reliant on short distance scales. Vacuum energy is Lorentz invariant. This is evident from the fact that under 3D spatial rotations and frame boosts, the vacuum energy, a scalar quantity does not alter. Further implications are that this ensures 1.) the energy density of the vacuum remains constant throughout the universe and 2.) the isotropic pressure of the stress-energy tensor is equal to the negative of its own vacuum density, $p_{vac} = \rho_{vac}$ (also evident form above calculations).

Experimental evidence is also present to support the above claim, notable being the Michelson-Morley light propagation and the independence of atomic properties.

Lorentz invariance tells us that the only stress energy (tensor component) consistent with it is:

$$T^{(\mathrm{vac})}_{\mu\nu} = -\rho_{\mathrm{vac}} g_{\mu\nu} \tag{3.1}$$

Additionally, the conservation of the stress-energy tensor, i.e., its covariant derivative, $\nabla_\mu T^{\mu\nu} = 0 \implies \rho_{vac} = constant$. The perfect fluid stress-energy tensor is given by:

$$T^{\mu\nu} = p g^{\mu\nu} + (p+\rho)u^\mu u^\nu \tag{3.2}$$

we also know from above, that the vacuum $\implies$ $p_{vac} = \rho_{vac}$ and so the corresponding equation of state parameter, i.e., the constant that tells us how the energy densities of components alter under expansion is:

$$w := \frac{p_{vac}}{\rho_{vac}} = -1. \tag{3.3}$$

For cosmology, this pressure does the work necessary to ensure that a constant energy density is maintained along with the expansion of space.

3.1 What must be explained in terms of Cosmology?

Precise measurements of the CMB have allowed us to confirm that the theory that the universe is a result of the expansion of a hot primordial soup is indeed correct and this precision allows to survey the energy content of the Universe.

Surprisingly, these results tell us that 2 components are required to explain this energy content:

- **Dark Matter**: Supported by multiple astrophysical and cosmological measurements, makes up $\frac{1}{4}$ of the obs. total energy density.

  Based on our presumptions, this component must cluster under gravitational attraction and so cannot cause an acceleration of the universal expansion. Hence, our focus falls on:

- **Dark Energy**: It makes up 70% of the obs. energy density. This is due to 2 facts:

  1.) The CMB data tell us that the universe is spatially flat $\implies$ total energy density must be critical, i.e., must describe that expansion rate of the universe approaches zero asymptotically. Therefore, The total energy density must contain more components other than just std. matter and dark matter.

  2.) Distance-redshift relation tests tell us that the expansion is accelerating. Additionally, the acceleration value is consistent for a spatially flat universe if the dark energy fluid has an equation of state parameter consistent with $w = 1$. So, our obs. that dark energy is 70% of the critical density $\implies$

$$\rho_{vac} \simeq (3 \times 10^{-3}\text{eV})^4 \tag{3.4}$$

However, these alone are not sufficient in concluding that dark energy must be a vacuum energy. For e.g., by utilizing slow varying scalar fields, ϕ^i, we could do away with the requirement of Lorentz invariance: If the k.e density of these fields is $K = \frac{1}{2}\mathcal{G}_{ij}(\phi)\partial_t\phi^i\partial_t\phi^j$ and if their scalar potential is $V(\phi)$, then:

$$\begin{aligned}
T^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^k)}\partial^\nu\phi^k + \mathcal{L}g^{\mu\nu} \\
\implies \rho = T^{00} &= \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^k)}\partial^0\phi^k + \mathcal{L}g^{00} \\
Using, \quad \mathcal{L} &= -\frac{1}{2}g^{\mu\nu}\mathcal{G}_{ij}(\phi)(\partial_\mu\phi^i)(\partial_\nu\phi^j) - V(\phi), \\
So, \quad \mathcal{L}_k &= -\frac{1}{2}g^{00}\mathcal{G}_{ij}(\phi)\dot{\phi}^i\dot{\phi}^j = K \\
\frac{\partial\mathcal{L}}{\partial(\partial_0\phi^k)} &= \frac{\partial\mathcal{K}}{\partial(\dot{\phi}^k)} = \mathcal{G}_{kj}(\phi)\dot{\phi}^j \\
\implies \rho &= (\mathcal{G}_{kj}\dot{\phi}^j)\dot{\phi}^k - (K - V)g^{00} \\
\implies \rho &= 2K - (K - V) = K + V \\
Similarly, \quad p = \frac{T_{11}}{g_{11}} &= \frac{G_{ij}(\partial_1\phi^i)(\partial_1\phi^j) + g_{11}\mathcal{L}}{g_{11}} = K - V
\end{aligned}$$

So, we have the total energy density and pressure, $\rho = K + V$ and $p = K - V$, $\implies w = \frac{p}{\rho}$ is a monotonically increasing function of K/V which satisfies $-1 \leq w \leq 1$, and is close to -1 whenever V is much bigger than K. As this requires ϕ^i to vary quite slowly, it requires the potential to have a very slowly varying region along which it is close to constant.

Observationally, this region is indistinguishable from a C.C that is in the limit that the potential is perfectly flat, with fields with $K = 0$.

Although this generalization allows more theories, for obs. viewpoint, such complicated scalar models are unnecessary and he dark energy looks exactly like what would be predicted by a cosmological constant.

3.2 The 'Old' C.C Problem

The underlying problem: Vacuum energy can be computed and is massive compared to exp. data.

We need to calculate the energy of the ground state for a quantum field. Therefore, let us consider a massive scalar field, h, and could be a Higgs field or serve as a proxy for the field on any elementary particle.

Consider the action:

$$S = -\int d^4x\sqrt{-g}\left[V(h) + \frac{1}{2}g^{\mu\nu}\partial_\mu h\,\partial_\nu h\right] \tag{3.5}$$

where the scalar potential could be:

$$V(h) = V_0 + \frac{1}{2}V_2h^2 + \frac{1}{4}V_4h^4 \tag{3.6}$$

where $V_0, V_2, V_4,$, etc. are real constants. Now, if we constrict ourselves to a static metric, $ds^2 = -dt^2 + g_{ij}dx^idx^j$ (static as no time dependence), time-translation invariance of said metric will give us a conserved energy, due to Noether's theorem. So, $g_{00} = g^{00} = -1$ and $g_{i0} = g^{i0} = g_{0i} = g^{0i} = 0$
$\implies g^{\mu\nu}\partial_\mu h\,\partial_\nu h = -(\partial_0 h)^2 + g^{ij}\partial_i h\,\partial_j h.$

Therefore, the Lagrangian density becomes:

$$\mathcal{L} = -\sqrt{-g}\left[V(h) - \frac{1}{2}(\partial_0 h)^2 + \frac{1}{2}g^{ij}\partial_i h\,\partial_j h\right] \tag{3.7}$$

As the Hamiltonian is given by: $\mathcal{H} = \pi\partial_t h - \mathcal{L}$, we also need the canonical momentum:

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_t h)} = \sqrt{-g}\partial_t h.$$

. Finally, we obtain:

$$\mathcal{H} = \sqrt{-g}\left[V(h) + \frac{1}{2}(\partial_t h)^2 + \frac{1}{2}g^{ij}\partial_i h\,\partial_j h\right]. \tag{3.8}$$

This is bounded below if V is bounded below and, i,j run over spatial indices only.
If we want to compute the vacuum energy semi-classically, for weak interactions, we need to expand the action about its lowest-energy classical solution. Our $\mathcal{H}$ is minimized for constant fields,

$$V'(\bar{h}) = (V_2 + V_4\bar{h}^2)\bar{h} = 0. \tag{3.9}$$

To get the classical vacuum (energy) let us consider $V_2 \geq 0$ for which it implies the energy is minimized at $H(\bar{h}) = V_0$ when $\dot{h} = 0$ while the minimum is instead at $\dot{h}^2 = -V_2/V_4$ if $V_2 < 0$. Here, energy density is

$$H(\bar{h}) = V_0 - \frac{(V_2)^2}{4V_4} \tag{3.10}$$

Next, in order to compute the leading order quantum corrections, we need to expand the action to quadratic order in $\hat{h}$ where (in the interaction picture)

$$\hat{h}(x) := h(x) - \bar{h} = \sum_k \left[u_k(x)a_k + u_k^*(x)a_k^*\right], \tag{3.11}$$

where

$$\left[a_k, a_l^\dagger\right] = \delta_{kl}, \tag{3.12}$$

are creation and annihilation operators and $u_k(x)$ are the normalized basis of energy eigenfunctions satisfying the linearized field equation $\left[-\Box + V''(\bar{h})\right] u_k = 0$ and $\partial_t u_k = -i\omega_k u_k$ with $\omega_k > 0$.

In terms of these modes the total energy takes the familiar harmonic-oscillator form

$$\begin{aligned}\hat{H} = \int d^3x\, \mathcal{H} &= \int d^3x \sqrt{-g} V(\bar{h}) + \frac{1}{2}\sum_k \omega_k (a_k a_k^\dagger + a_k^\dagger a_k) \\ &= E_0 + \sum_k \omega_k a_k^\dagger a_k.\end{aligned} \tag{3.13}$$

Here, we use $\frac{1}{2}(a_k a_k^\dagger + a_k^\dagger a_k) = a_k^\dagger a_k + \frac{1}{2}$ which comes from eq. 3.12 and defined

$$E_0 = \frac{1}{2}\sum_k \omega_k \mathcal{V} + V(\bar{h})\mathcal{V}, \tag{3.14}$$

where $\mathcal{V} := \int d^3x \sqrt{-g}$ is the total volume of space.

Applying $\hat{H}$ to the generic state, $|\Psi\rangle = \prod_k (a_k^*)^{n_k}|\Omega\rangle$ with $n_k = 0, 1, 2, \ldots$, we find $\hat{H}|\Psi\rangle = E_\Psi|\Psi\rangle$, with eigenvalue

$$E_\Psi = E_0 + \sum_k n_k \omega_k. \tag{3.15}$$

The vacuum is the eigenstate with the lowest energy, corresponding to $|\Omega\rangle$, for which $n_k = 0\ \forall k$. Its energy therefore is E_0, and so the prediction for $\hat{h}$'s contribution to the vacuum energy density is

$$\rho_{\text{vac}} = \frac{E_0}{\mathcal{V}} = V(\bar{h}) + \frac{1}{2\mathcal{V}}\sum_k \omega_k. \tag{3.16}$$

However, let us consider, for example, the following case: in flat space k is simply 3-momentum, and $\omega_k = \sqrt{\vec{k}^2 + m^2}$ where $m^2 = V''(\bar{h})$ is the particle mass, and $(2\pi)^3 \sum_k \omega_k = \mathcal{V} \int d^3k\, \omega_k$, which diverges quartically as $k \to \infty$. If the integration is cut off so $k \leq \Lambda \gg m$, then

$$\rho_{\text{vac}} = V(\bar{h}) + c_0\Lambda^4 + c_{2a}\Lambda^2 m^2 + c_{4a} m^4 + \cdots, \tag{3.17}$$

with calculable dimensionless coefficients c_i.[10] To this we must add a similar contribution from all other particles in Nature, leading to

$$\rho_{\text{vac}} = V(\bar{h}) + \sum_a \left(c_{0a}\Lambda^4 + c_{2a}\Lambda^2 m_a^2 + c_{4a} m_a^4 + \cdots\right). \tag{3.18}$$

The precise coefficients of the various powers of Λ are altered by the sum across particle types, but the powers of Λ that manifest are generally unaffected.[11]

It is frequently stated that we should select the greatest scale for Λ that we believe would be appropriate for this type of computation. Since a complete theory of quantum gravity is needed above this scale, and simple quantum field theory may not be sufficient for it, making the result less sensitive to significantly smaller distances, this could be interpreted as the "Planck Mass", $M_{\text{P}} := G_N^{-1/2} \approx 10^{19}$ GeV. And, thus, we get the huge inconsistency:

$$\frac{\rho_{\text{vac}}(\text{theory})}{\rho_{\text{vac}}(\text{observed})} \approx \frac{1}{(4\pi)^2}\left(\frac{10^{19}\text{ GeV}}{10^{-3}\text{ eV}}\right)^4 \approx 10^{122}. \tag{3.19}$$

4 Origins of Quintessence

4.1 Experimental Observations: Dark Energy

The Supernova Cosmology Project studied 42 Type Ia Supernovae and measured the values of the mass density, Ω_M, and the cosmological-constant energy density, Ω_Λ. By considering a flat cosmological background, i.e., $\Omega_M + \Omega_\Lambda = 1$, they measured:

$\Omega_M^{flat} = 0.28^{+0.09}_{-0.08}(1\sigma \text{ statistical})^{+0.05}_{-0.04}$ (identified systematics). Using this, the deceleration parameter, $q_0 = \Omega_M/2 - \Omega_\Lambda$, was found to be $q_0 = -0.58$ which implies that the universe is undergoing an accelerated expansion. Further, this indicated that the energy density was very small relative to the vacuum density, implying a vacuum energy domination. Further, the Supernova Cosmology Project and the High-Z Supernova Team also found, $\Omega_\Lambda = 0.75^{+0.06}_{-0.07}$(statistical)$\pm 0.032$(identified systematics).

The analysis and data above reflect the fact that matter and vacuum energy play diametric roles on the expansion of the universe, with matter causing a deceleration while vacuum energy causing an acceleration. The data above also confirms the conjecture that approximately 70% of the universe is made up of a mysterious component known as **dark energy** - a hypothesis supported by alternative measurements such as the CMB and BAO.

For an FLRW universe, the Einstein equations become,

$$-4\pi G(3P + \rho) = \frac{\ddot{a}}{a} \tag{4.1}$$

and the continuity equation is,

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{4.2}$$

For dark energy, the perfect fluid encompassing the universe has a negative pressure component. Thus dark energy aligns with the fact that the dominating part of the energy of the universe has to be in a form: $\rho + 3P < 0$, to account for an expanding universe.

4.2 Equation of State Parameter

The inflationary model of the universe allows us to envision the universe as flat and seconds the hypothesis that the universe is undergoing an accelerated expansion. To understand this expansion, we need to investigate the matter content of the universe (which we have done in the previous section through experimental data) and in order to that, we need to study the relation between the pressure, P, and the energy density, ρ, of the fluid that encompasses the universe. This can be done by analyzing the **equation of state parameter**, defined as,

$$w \equiv \frac{P}{\rho} \tag{4.3}$$

This linear, constant equation of state works for all cosmological fluids. Under this parameterization, the continuity equation becomes:

$$\begin{aligned} \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \\ \implies \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + w\rho) = 0 \\ \implies \dot{\rho} = -3\frac{\dot{a}}{a}\rho(1+w) \end{aligned}$$

$$\implies \frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \tag{4.4}$$

$$\implies \rho \propto a^{-3(1+w)}, \tag{4.5}$$

.

This highlights how the equation of state dictates the dilution of the energy density and this holds for a dust filled universe, $P = 0$, and a radiation dominated universe, $P = \frac{\rho}{3} \implies w = \frac{1}{3}$.

We can identify dark energy using the equation of state parameter:

$$w_{DE} = \frac{P_{DE}}{\rho_{DE}} \quad \text{where,} \quad w_{DE} < -\frac{1}{3} \tag{4.6}$$

For the ΛCDM model, dark energy admits an equation of state parameter of $w = -1$ which gives us a corresponding energy density scale of $\rho \propto a^0$, which corresponds to a constant energy density $\rho_\Lambda \sim \frac{\Lambda}{8\pi G_4}$, i.e., we get a de Sitter universe. This is supported by the Supernova Cosmology Project and the High-Z Supernova Team, which found that combining the supernovae data with independent measurements of Ω_M from CMB and galaxy redshift distortion data for a flat universe, yields,

$w = -1^{+0..15}_{-0.20}$(statistical)$\pm$0.09(identified systematic).

The above equation of state is a general function of time. For a time-varying equation of state, we can use, $w(a) = w_0 + (1-a)w_a$, under the assumption w_0 and w_a are constants (as is done for phenomenological purposes). This correlates to early universe inflation, in which the dark energy equation of state parameter changes with time. The present day *Planck* data, the current energy density is $\rho_{DE,0} \sim 10^{-120} M_{Pl}^4$ and the corresponding present day equation of state parameter being,

$w_{DE,0} = -1.03 \pm 0.03$

Therefore, moving forward, we can go off on two tangents:

1) The cosmological constant, Λ is a sound assumption if the measured value of w continues to converge to an amount that is arbitrary close to -1.This is our standard ΛCDM model and here, we may consider future measurements to converge to the same value.

2) Alternatively, we can consider that the observational data will converge to a value nearly but not quite to -1. This correlates to a time varying dark energy condition. Our leading candidate here is quintessence.

4.3 Revisiting the Cosmological Constant

Einstein (1977) proposed the cosmological constant which, with $w = -1$, offers the simplest explanation to the existence of dark energy and is currently used by the ΛCDM model. Currently, there exists no solid reasoning so as to explain its existence and to explain why its present magnitude is so small, yet causing an acceleration in the universal expansion in the present epoch. The problems of the cosmological constant can be categorized as:

1. **The fine tuning problem**: The problem regarding how the magnitude of λ is so much further away from the Planck scale when defined by fundamental constants, i.e., the discrepancy between the 'massive' theoretical value and the 'tiny' observed value

2. **The coincidence problem**: The problem of how the expansion of the universe started to accelerate only recently, when densities of matter and dark energy are comparable i.e., since inflation, the universe has expanded by roughly $\approx 10^{28}$, however the scale factor has only increased by an order of 2: from 0.5 to approx. $1 \implies \rho_m$ and ρ_λ are comparable in present time, coincidentally.

5 Characteristics of Quintessence

5.1 The building blocks of Quintessence

In light of the problems highlighted above and to accommodate the newer results from the *Planck* experiment, we need to consider the auxiliary approach to dark energy, in which the cosmological constant disappears completely. Such an approach would constitute a dynamic equation of state parameter and so would also need to be compliant with the inflationary universe and thus, by extension, with rolling scalar fields (fields which roll along with their respective scalar potentials, $V(\phi)$). As such, we will use a canonical scalar field that is minimally coupled to gravity: 1) Being canonical ensures that no unwanted ghosts or Laplacian instabilities are generated, gradient instabilities are avoided and allows us to work on familiar grounds (like the Higgs field) 2) Minimal coupling ensures that simplicity is maintained; by being coupled only to gravity ensures equivalence principle holds and by extension compliance with GR and, does not modify the gravity in any way.

5.1.1 Quintessence and Inflation

Considering the scalar field to be slowly rolling along a potential $V(\phi)$ allows for it to account for the acceleration of the universe. In order to accommodate inflation further in the context we quintessence, we can use **quintessential inflation**.

5.2 Lagrangian Dynamics of Quintessence

For quintessence, we can consider the (total) action for non-relativistic matter whose pressure is solely dependent on pressure, i.e, $P = P(\rho)$. This is essentially the standard general relativity action with $\Lambda = 0 \implies S = S_g + S_m$. So,

$$S = \int \sqrt{-q}[\frac{1}{2}M_{pl}^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\phi - V(\phi)] \tag{5.1}$$

where $g = Det(g_{\mu\nu})$, M_{Pl} is the reduced Planck mass, R is the Ricci scalar and S_m is the matter action which we set to 0 by assuming that the non-relativistic matter does not have a direct coupling to the quintessence field, ϕ. Here, we also used the scalar field Lagrangian,

$$\mathcal{L}_\phi = \frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\phi - V(\phi) \tag{5.2}$$

Using the energy momentum tensor,

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}} \tag{5.3}$$

and then, to identify the energy density and pressure,

$$S = \int d^4x \sqrt{-g}\mathcal{L}_\phi$$

Using (2),

$$\sqrt{-g}\mathcal{L}_\phi = \sqrt{-g}\left(\frac{1}{2}g^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) - V(\phi)\right),$$

$$\frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}} = \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\mathcal{L}_\phi + \sqrt{-g}\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}}$$

Varying $\sqrt{-g}$ and $\mathcal{L}_\phi$,

$$\implies \frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\mathcal{L}_\phi + \sqrt{-g}\cdot\frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi).$$

$\implies T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\mathcal{L}_\phi$Finally, substituting back into (2) and using (1),

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial_\alpha\phi\partial^\alpha\phi) + g_{\mu\nu}V(\phi).$$

Now, we will compute T_{00} for ρ_ϕ & T_{ii} for p_ϕ

$$T_{00} = \partial_0\phi\partial_0\phi - \eta_{00}\mathcal{L}_\phi, \quad \mathcal{L}_\phi = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)$$

$$T_{ii} = \partial_i\phi\partial_i\phi - \eta_{ii}\mathcal{L}_\phi, \quad \mathcal{L}_\phi = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)$$

And using $T_{ii} = (\partial_i\phi)^2 + \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi),$

$$\sum_{i=1}^{3} T_{ii} = \sum_i\left[(\partial_i\phi)^2 + \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right] = \frac{3}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - 3V(\phi)$$

Finally, using $\rho_{phi} = T_{00}$ and $p_\phi = \frac{1}{3}T_{ii}$,

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}(\nabla\phi)^2 \tag{5.4}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) - \frac{1}{6}(\nabla\phi)^2 \tag{5.5}$$

The Compton wavelength of the field will be of the same order or larger than the Hubble scale, as late-time necessitates the use of a very light scalar field, therefore, the field is anticipated to be spatially smooth inside the Hubble scale, $\implies$ neglecting the spatial gradient terms (i.e., measure of how quickly the quantities change) of the energy density and pressure. We can further use the standard Friedmann equations with thsese quantities to figure the scale factor, $a(t)$, using the Hubble parameter, $\frac{\dot{a}}{a}$, and acceleration, $\ddot{a}$

We can now define the equation of state ratio/parameter, which normally varies with time:

$$w = p_\phi/\rho_\phi \tag{5.6}$$

Dynamical fields generally imply time-varying fields $\implies$ $w \neq$ constant. The equation of motion of concern for scalar fields in the Klein-Gordon Equation:

$$\ddot{\phi} + 3H\dot{\phi} = -dV/d\phi \tag{5.7}$$

and so is replaceable with the continuity equation.

For example, assuming a homogeneous scalar field, $\nabla\phi = 0$, let:

$$V = (\rho_\phi - p_\phi)/2 = \rho_\phi(1-w)/2 \tag{5.8}$$
$$K \equiv \dot{\phi}^2/2 = (\rho_\phi + p_\phi)/2 = \rho_\phi(1+w)/2. \tag{5.9}$$

Now, multiplying (6) by $\dot{\phi}$,

$$\left[\frac{\dot{\phi}^2}{2}\right]^{\cdot} + 6H\left[\frac{\dot{\phi}^2}{2}\right] = -\dot{V} \tag{5.10}$$

Now, using $\dot{\rho}_\phi = \frac{d}{dt}\left(\frac{\dot{\phi}^2}{2}\right) + \dot{V}$. using $\frac{d}{dt}\left(\frac{\dot{\phi}^2}{2}\right) = -\dot{V} - 6H\left(\frac{\dot{\phi}^2}{2}\right)$ from (9),

$$\dot{\rho}_\phi - \dot{V} + 3H(\rho_\phi + p_\phi) = -\dot{V} \tag{5.11}$$

Thus, we get an equation of state parameter of the form:

$$w \equiv \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}. \tag{5.12}$$

Further, if we use the observational data for cosmological expansion in order to derive the fluid equations of motion,

$$\rho_\phi(a) = \Omega_w \rho_c \exp\left\{3\int_a^1 d\ln a[1+w(a)]\right\} \tag{5.13}$$
$$\phi(a) = \int d\ln a H^{-1}\sqrt{\rho_\phi(a)[1+w(a)]} \tag{5.14}$$
$$V(a) = \rho_\phi(a)[1-w(a)]/2 \tag{5.15}$$
$$K(a) = \dot{\phi}^2/2 = \rho_\phi(a)[1+w(a)]/2. \tag{5.16}$$

where a denotes the acceleration parameter. Finally, to see that we can only partially study the scalar field physics at play in the $1+w \ll 1$ region, as ascertained by observational data, let us derive the time evolution equation of the scalar field,

$$\dot{\phi} = [\rho_\phi(1+w)]^{1/2} \lesssim HM_{\mathrm{P}}(1+w)^{1/2}, \tag{5.17}$$

Tracker Solutions Tracker fields have equations of motions which admit attractor-esque solutions. A state or group of states that a system naturally progresses toward, independent of a broad variety of initial conditions, is known as an attractor. These kinds of solutions begin in the basin of attraction and, as the system evolves, move closer to the attractor. In the system's phase space, the solutions themselves are stable to small perturbations and insensitive to initial conditions. An analogous evolutionary path rapidly develops from an eclectic mix of initial conditions for tracker fields. They resolve the coincidence problem by avoiding the early universe fine-tuning requirement necessary to bring them into alignment presently because the evolution of the scalar field "tracks" the dominant background density (matter or radiation) for a considerable amount of time while remaining subdominant. This is because the energy density of the scalar field typically decreases more slowly than the matter density.

5.3 Classification of Quintessence Models

5.3.1 Classification based on general dynamical behavior

There are four possible behaviors for scalar fields at any particular time period: fast, slow, steady, oscillatory.

1. **Fast Roll**: When the quintessence fields's kinetic energy is greater than its potential energy and thus $w > 0$, we have the fast roll condition.

   Kination - fast roll period - is inherent to tracker models that follow attractor trajectories in their dynamics, meaning that the universe's dominating energy density component determines their equation of state at particular epochs. The scalar field's energy density can be quickly reduced by the fast roll from an initial early universe value close to the "natural" Planck scale to a considerably smaller value that will be appropriate for the observed present energy density. The fine tuning problem of the cosmological constant can be addressed by a wide range of initial circumstances, or "basis of attraction," that can provide a tolerable present energy density for specific forms of the potential due to the attractor solution for the dynamics. The coincidence problem is not entirely resolved due to the fact that in order to dominate the energy density and produce acceleration, the field must exit both the tracking and fast roll regimes. Specifically, tracking fields are no longer thought to be the leading candidates for explaining the acceleration since they struggle to reach equations of state $w \lesssim -0.7$ in contest with observations.

2. **Slow Roll**: Slow roll quintessence occurs when the kinetic energy is significantly lower than the potential energy, leading to an equation of state parameter of $w \approx -1$.

   This only leads to acceleration of the expansion if the dark energy equally dominates the energy density. If nothing else changed, it would eventually take over the universe because the field is almost frozen, the dark energy density is almost constant, and matter and radiation are fast decreasing from expansion. $\frac{V'}{V}$, a common slow roll parameter for inflation, may not have a small value for a field that we consider to be slowly rolling, $w \approx -1$. Without the tracker models' basin of attraction, quintessence modes that always have the potential dominant over the kinetic term face the same fine tuning and coincidence constraints as the cosmological constant.

3. **Steady Roll**: Going off the advantages and disadvantages of fast and slow roll, we require both behaviors to be present for a successful quintessence model.

   Throughout its history, this field has undergone both fast and slow roll phases,employing a linear potential. The Klein-Gordon equation of motion nevertheless appears to have a constant right-hand side in the linear potential model, and the dynamics for a considerable amount of time remain quite close to the line where the field acceleration $\ddot{\phi}$ is zero. This model is the most straightforward extension of the cosmological constant. Due to Hubble friction, it begins generically in a frozen condition resembling a cosmic constant, then thaws and rolls down the potential. But instead of accelerating expansion, the field rolls into a region where the potential goes negative since the potential has no minimum. This results in a collapsing universe implying a finite future.

4. **Oscillatory Roll**: In re-normalizable field theories, $V(\phi) \sim \phi^n$ is a common potential with a minimum for an even n. The field will first have a typical rolling stage before reaching the minimum and oscillating around it. If the oscillation period is significantly less than the Hubble time, the effective equation of state becomes: $w = \frac{n-2}{n+2}$.

   The field resembles like radiation for a quartic potential and like nonrelativistic matter for a quadratic potential. Pseudo-Nambu Goldstone bosons (PNGB), or the axion, are an interesting example of such a field. In contrast to an ad hoc $V(\phi)$ that might be written down but later acquire a non-zero ground state and distortion of its shape, PNGB potentials are also radiatively stable against quantum corrections if we consider them during the regime when they are still rolling rather than oscillating. However, this acceleration will eventually decay as the field evolves to its oscillatory, matter-like phase. Thus, the physics of such pseudoscalar fields offers a possibility for a fundamental, high-energy origin rather than merely a low energy effective potential. The PNGB potential resembles: $V(\phi) = V_0[1 + cos(\phi/f)]$, where f is a symmetry energy scale.

5.3.2 Classification based on potentials

1. **Exponential Quintessence**: For exponential quintessence, we consider a single canonical field ϕ and consider an exponential potential,

$$V(\phi) = V_0 e^{-\lambda\phi/M_{\rm pl}} \tag{5.18}$$

   where $V(\phi)$ is the potential energy of the scalar field, V_0 and λ (lambda) are constant parameters and $M_{\rm pl}$ is the reduced Planck mass. The advantage of such a model is that it naturally leads to attractor solutions. The parameters of the model must eventually lead the scaling behavior to break down and the field to become dominant, simulating a cosmological constant (Λ), in order to produce the currently observed late-time cosmic acceleration.

   As such, we can introduce a consatnt field value, $\phi_* : e^{\lambda\phi_*} \equiv \frac{V_0}{H_0^2}, \tilde{V} = e^{-\lambda(\phi-\phi_*)}$. Thus the field can be redefined to be $\phi \longrightarrow \phi + \phi_*$ to absorb the constant and so up to a constant, the field's value will always be known. Now, the potential is only characterized by $\lambda, \tilde{V} = e^{-\lambda\phi}$.

   However, without adding additional complexities, such as coupling—which introduces an interaction between the quintessence field and dark matter—and modified potential—which uses a double exponential potential or combines the exponential term with a power-law or another term to better fit the observed dynamics—models that only use a single, uncoupled exponential potential frequently struggle to satisfy all observational constraints, such as CMB and BAO data.

2. **Hilltop Quintessence**: In hilltop quintessence, the defining fact is that the near a local maximum (or "hilltop") of its potential energy function $V(\phi)$, the scalar field (ϕ) is presently rolling very slowly.

   The potential $V(\phi)$ can be roughly expanded using a concave function near its maximum, ϕ_*:

$$V(\phi) \approx V_0 - \frac{1}{2}m^2(\phi - \phi_*)^2 + \dots \tag{5.19}$$

   Specifically, we can use,

$$V(\phi) = V_0 \left(1 - \frac{\kappa^2}{2}\phi^2\right), \quad V_0, \kappa > 0 \tag{5.20}$$

   When we implicitly restrict ourselves to the field range for which $V > 0$, the potential depends on 2 parameters, $\tilde{V} = \frac{V_0}{H_0^2}, \kappa$.
   Alternatively, we can use the previously mentioned PNGB potential,

$$V(\phi) = \mu^4[1 + cos(\frac{\phi}{f})] \tag{5.21}$$

   where μ and f are constants.

   Hilltop quintessence are a subset of thawing models. During the early universe, The initial value of the field ϕ is infinitesimally close to the potential hilltop. Near the maximum, the potential is quite flat, therefore Hubble friction greatly suppresses the motion of the field. Thus, the equation of state parameter is very close to the value of the cosmological constant, $w_\phi \approx -1$, and the kinetic energy is negligible $(\frac{1}{2}\dot{\phi}^2 \ll V(\phi))$. It is similar to a cosmological constant in that the dark energy density is almost constant. During the late universe, i.e, the thawing period, the Hubble parameter falls as the universe expands and matter density decreases. The field eventually "thaws" or starts to roll away from the hilltop as a result of decreased kinetic friction. Thus, in relation to the potential energy, the kinetic energy begins to rise. The equation of state parameter w_ϕ starts to move away from -1 and toward greater values.

   In agreement with the data favoring thawing models over freezing models, where w is more negative than the current value, hilltop quintessence naturally provides a mechanism to maintain $w \approx -1$ for a long time—while the field is "frozen" near the hilltop—and only begin to diverge in the recent past.

6 Coda and Vista

Understanding the nature of dark energy and the cosmological constant is still one of the most important problems facing current cosmology. Using quintessence, a dynamic scalar field, as a theoretical and observationally plausible solution, we tried to review this issue here. We examined the cosmological constant problems and saw how they are addressed by various quintessence models. Understanding the dark energy field's evolution is essential to comprehending this phenomenon. Observationally, the quintessence landscape is limited but encouraging. The Supernova Project, CMB, and BAO data currently available strongly support an equation of state that is quite similar to a cosmological constant, with constraints such as $w(0) < -0.964$. For a large portion of the cosmological existence, this prompts viable quintessence theories to behave almost exactly like ΛCDM. However, quintessence's dynamic nature leaves behind faint, distinctive traces that form the foundation for its final identification or rejection.

A multifaceted strategy aimed at capturing its fluid and fleeting nature is needed to differentiate quintessence from ΛCDM. Precise measurement of w is the target of upcoming projects like DESI, the Vera C. Rubin Observatory, and the Euclid space telescope. These also aim to identify the shifts in matter domination era between ΛCDM and a quintessence model and the shift in matter and dark energy equality. The proposed early "kination" phase of quintessence, the fast transition to $w_\phi \approx -1$ from $w_\phi \approx +1$ during radiation domination may leave traces on the stochastic gravitational wave background or the CMB and thus also require further probing.

The "freezing" of scalar fields during the radiation and matter eras as a result of strong Hubble friction successfully simulates moduli stabilization without the need for a vacuum, which is a very difficult aspect of model creation. This implies that fields might have been fixed on a possible slope for the majority of cosmic history until just recently starting to roll. For string model construction, figuring out how such fields couple to matter is an open task.

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Approval

The internship report titled **"Quintessence"** submitted by **M. Fahim Hoque**, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfillment of the requirements of the internship program.

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Supervisor

Ahmed Rakin Kamal
Department of Theoretical Physics and Astrophysics, Faculty of Science, Masaryk University, 611 37 Brno, Czech Republic