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'Conformal Field Theory'

Report prepared by: Fabiha Noshin

Supervised by: Samanta Saha

1. Introduction

Conformal transformation is accounted for as the transformation of the coordinates in such a way that the overall metric, which is not dynamical, remains invariant up to an overall constant of the initial metric. Conformal field theories are the field theories that remain invariant under these conformal transformations of the background manifold's metric where these field theories are defined. In other words, conformal transformation is a change of coordinates at its heart, where it is a smooth transformation of the metric such that the metric of the manifold where the field is defined is changed in a way that the new metric is an overall scale factor of the original metric, from which the notion of 'preservation of angles' emerges.

In this report, the notion of conformal symmetry has been explored: how conformal symmetries in different dimensions are elements of the conformal group ($d > 2$ and $d = 2$). Then different generators for the conformal groups have been derived along with their conformal algebra. After that, the physical consequences of the different conformal symmetries on the classical field have been seen. An example has been explored: $2d$ classical free scalar field in Euclidean space. Lastly, the report ends with ideas of further exploration of the route, that is, looking into the quantum conformal field theories.

2. Conformal Transformations, Groups and their Algebras

Let M be a manifold and $M = \mathbb{R}^{(p,q)}$ where $p+q = d$ ($p, q \geq 0; p, q \in \mathbb{Z}$), where $g_{\mu\nu} = \text{diag}(\underbrace{1, 1, \ldots, 1}_{p}, \underbrace{-1, -1, \ldots, -1}_{q})$.

Here a global coordinate can be defined on this manifold. Let an infinitesimal conformal transformation of the coordinate: $x^\mu \to x^\mu + \epsilon^\mu(x)$ where the global metric changes as: $g'_{\mu\nu}(x) = g_{\mu\nu}(x) + \frac{\partial\epsilon_\nu}{\partial x^\mu} + \frac{\partial\epsilon_\mu}{\partial x^\nu} + O(\epsilon^2)$. Now in order for the new metric to be an overall constant of the initial metric, $g'_{\mu\nu}(x) = \Omega(x)g_{\mu\nu}(x)$ implies $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \alpha(x)g_{\mu\nu}$. That is, $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \alpha(x)g_{\mu\nu}$.

Now, $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \alpha(x)g_{\mu\nu}$

$\Rightarrow (\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu)g^{\mu\nu} = \alpha(x)g_{\mu\nu}g^{\mu\nu}$

$\Rightarrow \partial_\mu\epsilon_\nu g^{\mu\nu} + \partial_\nu\epsilon_\mu g^{\mu\nu} = \alpha(x)g_{\mu\nu}g^{\mu\nu}$

$\Rightarrow \partial_\mu\epsilon^\mu + \partial_\nu\epsilon^\nu = \alpha(x)d \quad [d = p + q]$

$\Rightarrow 2(d.\epsilon) = \alpha(x)d$

$\Rightarrow \alpha(x) = \dfrac{2(d.\epsilon)}{d}$

Now ignoring terms from $O(\epsilon^2)$ implies,

$$\Omega(x) = 1 + \frac{2(d.\epsilon)}{d} \tag{1}$$

That is, $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)g_{\mu\nu}$

$$\Rightarrow \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)g_{\mu\nu} \tag{*}$$

Taking divergence of the above equation yields,

$\Rightarrow \partial^\mu\partial_\mu\epsilon_\nu + \partial^\mu\partial_\nu\epsilon_\mu = \partial^\mu \left\{\frac{2}{d}(\partial \cdot \epsilon)g_{\mu\nu}\right\}$

$\Rightarrow \partial^\mu\partial_\mu\epsilon_\nu + \partial_\nu\left(\partial^\mu\epsilon_\mu\right) = \frac{2}{d}\partial_\nu(\partial \cdot \epsilon)$ [partial derivatives commute in flat spacetime]

$\Rightarrow \Box\epsilon_\nu = \frac{2}{d}\partial_\nu(\partial \cdot \epsilon) - \partial_\nu(\partial \cdot \epsilon)$

$$\Rightarrow \Box\epsilon_\nu = \left(\frac{2}{d} - 1\right)\partial_\nu(\partial \cdot \epsilon) \tag{**}$$

Now taking $\Box$ on equation (**) yields $\Rightarrow$

$\Box(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) = \frac{2}{d}\Box(\partial \cdot \epsilon)g_{\mu\nu}$

$\Rightarrow \partial_\mu \Box \epsilon_\nu + \partial_\nu \Box \epsilon_\mu = \frac{2}{d}\Box(\partial\cdot\epsilon)g_{\mu\nu}$
$\Rightarrow \partial_\mu\left\{\left(\frac{2}{d}-1\right)\partial_\nu(\partial\cdot\epsilon)\right\} + \partial_\nu\left\{\left(\frac{2}{d}-1\right)\partial_\mu(\partial\cdot\epsilon)\right\} = \frac{2}{d}\Box(\partial\cdot\epsilon)g_{\mu\nu}$
$\Rightarrow \left(\frac{2}{d}-1\right)\partial_\mu\partial_\nu(\partial\cdot\epsilon) + \left(\frac{2}{d}-1\right)\partial_\mu\partial_\nu(\partial\cdot\epsilon) = \frac{2}{d}\Box(\partial\cdot\epsilon)g_{\mu\nu}$
$\Rightarrow 2\partial_\mu\partial_\nu(\partial\cdot\epsilon)\left(\frac{2}{d}-1\right) = \frac{2}{d}\Box(\partial\cdot\epsilon)g_{\mu\nu}$
$\Rightarrow \frac{2}{d}\Box(\partial\cdot\epsilon)g_{\mu\nu} - 2\partial_\mu\partial_\nu(\partial\cdot\epsilon)\left(\frac{2}{d}-1\right) = 0$

$$\Rightarrow \{\Box g_{\mu\nu} + (d-2)\partial_\mu\partial_\nu\}(\partial\cdot\epsilon) = 0 \tag{2}$$

Consequences of Equations (1) and (2):
From equation 2, $\big(\Box g_{\mu\nu} + (d-2)\,\partial_\mu\partial_\nu\big)(\partial.\epsilon) = 0$ implies the third derivative of ϵ has to vanish. Therefore, the general solution for ϵ could be: $\epsilon_\mu(x) = a_\mu + B_{\mu\nu}x^\nu + C_{\mu\nu\rho}x^\nu x^\rho$ where a_μ, $B_{\mu\nu}$, and $C_{\mu\nu\rho}$ are constants.

- If $\epsilon_\mu(x) = a_\mu$, then all derivatives vanish, and hence this is a valid solution to equation (2).
- If $\epsilon_\mu(x) = B_{\mu\nu}x^\nu$, then,

$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}(\partial.\epsilon)g_{\mu\nu}$ (equation 1),
implies , $B_{\mu\nu} + B_{\nu\mu} = \frac{2}{d}(\partial.\epsilon)g_{\mu\nu} \quad\Rightarrow\quad B_{\mu\nu} + B_{\nu\mu} = \frac{2}{d}\,\mathrm{Tr}(B)\,g_{\mu\nu}$. Decomposing $B_{\mu\nu}$ into symmetric and antisymmetric components,

$$\begin{aligned} B_{\mu\nu} &= \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu}) + \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu}) \\ &= \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu}) + \frac{1}{2}\cdot\frac{2}{d}\,\mathrm{Tr}(B)\,g_{\mu\nu} \\ &= \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu}) + \frac{1}{d}\,\mathrm{Tr}(B)\,g_{\mu\nu}. \end{aligned}$$

From the antisymmetric part, $\epsilon^\mu(x) = w^\mu_\nu x^\nu, \quad w_{\mu\nu} = -w_{\nu\mu}$ [Lorentz rotations and boosts]
From the symmetric part, $\epsilon^\mu(x) = \lambda x^\mu$ [λ is the trace of B]

- Quadratic: if $\epsilon_\mu(x) = C_{\mu\nu\rho}x^\nu x^\rho$

$\partial_\alpha\epsilon_\beta = 2C_{\beta\alpha\rho}x^\rho$ that is, $\partial_\alpha\epsilon_\alpha = 2C_{\alpha\alpha\rho}x^\rho = 2(\partial.\epsilon)$
From equation 1, $\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = \frac{2}{d}(\partial.\epsilon)$
$\Rightarrow\ 2(C_{\mu\nu\rho} + C_{\nu\mu\rho})x^\rho = \frac{2}{d}(\partial.\epsilon)$
$\Rightarrow (C_{\mu\nu\rho} + C_{\nu\mu\rho})x^\rho = \frac{2}{d}C_{\alpha\alpha\rho}x^\rho g_{\mu\nu}$
This implies, $C_{\mu\nu\rho} = b_\mu g_{\nu\rho} - g_{\mu\nu}b_\rho - g_{\mu\rho}b_\nu$ [b_μ is constant]
$\epsilon_\mu(x) = (b_\mu g_{\nu\rho} - g_{\mu\nu}b_\rho - g_{\mu\rho}b_\nu)x^\nu x^\rho$
$= b_\mu g_{\nu\rho}x^\nu x^\rho - g_{\mu\nu}b_\rho x^\nu x^\rho - g_{\mu\rho}b_\nu x^\nu x^\rho$
$= b_\mu x^2 - 2x_\mu(b\cdot x)$
$\epsilon^\mu(x) = b^\mu x^2 - 2x^\mu(b\cdot x) \quad [g_{\mu\nu} = \eta_{\mu\nu}]$
Therefore, the general solution for the conformal coordinate transformation,

$$\epsilon^\mu(x) = a^\mu + w^\mu_v x^v + \lambda x^\mu + b^\mu x^2 - 2x^\mu(b\cdot x) \tag{3}$$

2.1. For $d \geq 3$:

Infinitesimal conformal transformations: $x^\mu \longrightarrow x^\mu + \epsilon^\mu(x), \quad x \in \mathbb{R}^{p,q}$
In general,$f(x+\epsilon) = f(x) + \epsilon^\mu\frac{\partial f}{\partial x^\mu} + O(\epsilon^2)$ If $f(x+\epsilon)$, $f(x)$ are coordinates then $\epsilon^\mu\partial_\mu$ is the vector field that generates the transformation and δf(change of any function) $= \epsilon^\mu\frac{\partial f}{\partial x^\mu}$ Therefore the required generators of the group in this context is $V = \epsilon^\mu\partial_\mu$.

From equation 3, $\epsilon^\mu(x) = a^\mu + w^\mu{}_\nu x^\nu + \lambda x^\mu + b_\mu x^2 - 2x^\mu(b\cdot x)$
When $\epsilon^\mu = a^\mu$, $V = a^\mu\partial_\mu \longrightarrow$ generator $\quad P_\mu = \partial_\mu$
When $\epsilon^\mu(x) = w^\mu_\nu x^\nu = g_{\alpha\nu} w^{\mu\alpha} x^\nu = g_{\alpha\nu} g^{\beta\nu} w^{\mu\alpha} x_\beta \Big|^{g=\eta}_{\text{that is, for minkowski spacetime}}$

$$V = w^{\mu\alpha} x_\alpha \partial_\mu$$
$$= \tfrac{1}{2}(w^{\mu\alpha} - w^{\alpha\mu}) x_\alpha \partial_\mu$$
$$= \tfrac{1}{2}\left\{w^{\mu\alpha} x_\alpha \partial_\mu - w^{\alpha\mu} x_\alpha \partial_\mu\right\}$$
$$= \tfrac{1}{2}\left\{w^{\mu\alpha} x_\alpha \partial_\mu - w^{\mu\alpha} x_\mu \partial_\alpha\right\} \quad (\alpha \leftrightarrow \mu)$$
$$= \tfrac{1}{2} w^{\mu\alpha}(x_\alpha\partial_\mu - x_\mu\partial_\alpha)$$

That is, $M_{\alpha\mu} = x_\alpha\partial_\mu - x_\mu\partial_\alpha$
When $\epsilon^\mu(x) = \lambda x^\mu$ $V = \lambda x^\mu\partial_\mu$. That is, $D = x^\mu\partial_\mu$
When $\epsilon^\mu(x) = b^\mu x^2 - 2x^\mu(b\cdot x)$,

$$V = \epsilon^\mu\partial_\mu = \left\{b^\mu x^2 - 2x^\mu(b\cdot x)\right\}\partial_\mu$$
$$= \left(b^\mu x^\alpha x^\beta \eta_{\alpha\beta} - 2x^\mu b^\alpha x^\beta \eta_{\alpha\beta}\right)\partial_\mu$$
$$= b^\mu x^\alpha x^\beta \eta_{\alpha\beta}\,\partial_\mu - 2x^\mu b^\alpha x^\beta \eta_{\alpha\beta}\,\partial_\mu$$
$$= b^\mu \partial_\mu\, x^\alpha x^\beta \eta_{\alpha\beta} - 2x^\mu \partial_\mu\, b^\alpha x^\beta \eta_{\alpha\beta}$$
$$= b^\mu \partial_\mu\, x^\alpha x^\beta \eta_{\alpha\beta} - 2x^\alpha \partial_\alpha b^\mu x^\beta \eta_{\mu\beta} \quad [\alpha \leftrightarrow \mu \text{ in the last term}]$$
$$= b^\mu x^2 \partial_\mu - 2b^\mu (x\cdot\partial) x_\mu$$
$$= b^\mu\left\{x^2\partial_\mu - 2(x\cdot\partial)x_\mu\right\}$$

That is, $K_\mu = x^2\partial_\mu - 2(x\cdot\partial)x_\mu$

Here, the conformal algebra between generators are the corresponding Lie brackets between the generators when $d \geq 3$:

$$\begin{aligned}
[P_\mu, P_\nu] &= [\partial_\mu, \partial_\nu] \\
&= \partial_\mu\partial_\nu - \partial_\nu\partial_\mu \\
&= 0 \quad \text{(commutes in flat spacetime)}
\end{aligned}$$

Since in this purpose of flat spacetime,it implies that the translation operations along different directions commute with each other.

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= [(x_\mu\partial_\nu - x_\nu\partial_\mu), (x_\rho\partial_\sigma - x_\sigma\partial_\rho)] \\
&= (x_\mu\partial_\nu - x_\nu\partial_\mu)(x_\rho\partial_\sigma - x_\sigma\partial_\rho) - (x_\rho\partial_\sigma - x_\sigma\partial_\rho)(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
&= x_\mu\partial_\nu(x_\rho\partial_\sigma - x_\sigma\partial_\rho) - x_\nu\partial_\mu(x_\rho\partial_\sigma - x_\sigma\partial_\rho) \\
&\quad - x_\rho\partial_\sigma(x_\mu\partial_\nu - x_\nu\partial_\mu) + x_\sigma\partial_\rho(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
&= x_\mu\eta_{\nu\rho}\partial_\sigma - x_\mu\eta_{\nu\sigma}\partial_\rho - x_\nu\eta_{\mu\rho}\partial_\sigma + x_\nu\eta_{\mu\sigma}\partial_\rho \\
&\quad - x_\rho\eta_{\sigma\mu}\partial_\nu + x_\rho\eta_{\sigma\nu}\partial_\mu + x_\sigma\eta_{\rho\mu}\partial_\nu - x_\sigma\eta_{\rho\nu}\partial_\mu \\
&= \eta_{\nu\rho}(x_\mu\partial_\sigma - x_\sigma\partial_\mu) + \eta_{\nu\sigma}(x_\rho\partial_\mu - x_\mu\partial_\rho) \\
&\quad + \eta_{\mu\rho}(x_\sigma\partial_\nu - x_\nu\partial_\sigma) + \eta_{\mu\sigma}(x_\nu\partial_\rho - x_\rho\partial_\nu) \quad [\eta_{\mu\nu} = \eta_{\nu\mu}] \\
&= \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\rho\mu} + \eta_{\mu\rho}M_{\sigma\nu} + \eta_{\mu\sigma}M_{\nu\rho}
\end{aligned}$$

These $M_{\mu\nu}$ is the required generator of the Lorentz group which generates the rotation and boosts in the

Lorentz spacetime.

$$\begin{aligned}[D,D] &= [x^\nu\partial_\nu, x^\mu\partial_\mu]\\ &= x^\nu\partial_\nu(x^\mu\partial_\mu) - x^\mu\partial_\mu(x^\nu\partial_\nu)\\ &= x^\nu\eta_{\nu\mu}\partial^\mu - x^\mu\eta_{\mu\nu}\partial^\nu\\ &= x^\nu\eta_{\nu\mu}\partial^\mu - x^\nu\eta_{\nu\mu}\partial^\mu \quad \text{(renaming indices } \mu\leftrightarrow\nu)\\ &= 0\end{aligned}$$

Dilations are essentially scalar transformations of the coordinates, which intuitively commutes with each other. Thus dilation generators along different directions of flatspacetime commutes.

$$\begin{aligned}[K_\mu, K_\nu] &= \left[x^2\partial_\mu - 2(x\cdot\partial)x_\mu, x^2\partial_\nu - 2(x\cdot\partial)x_\nu\right]\\ &= \left[x^2\partial_\mu, x^2\partial_\nu - 2(x\cdot\partial)x_\nu\right] - \left[2(x\cdot\partial)x_\mu, x^2\partial_\nu - 2(x\cdot\partial)x_\nu\right]\\ &= \left[x^2\partial_\mu, x^2\partial_\nu\right] - \left[x^2\partial_\mu, 2(x\cdot\partial)x_\nu\right] - \left[2(x\cdot\partial)x_\mu, x^2\partial_\nu\right]\\ &\quad + \left[2(x\cdot\partial)x_\mu, 2(x\cdot\partial)x_\nu\right]\\ &= 0\end{aligned}$$

In summary :

$$\begin{aligned}&[P_\mu, P_\nu] = 0\\ &[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\rho\mu} + \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\rho\nu}\\ &[D,D] = 0\\ &[K_\mu, K_\nu] = 0\\ &[P_\mu, D] = P_\mu\\ &[D, K_\nu] = K_\nu\\ &[M_{\mu\nu}, K_\rho] = \eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu\\ &[P_\mu, K_\nu] = 2\big(M_{\mu\nu} - \eta_{\mu\nu}D\big)\\ &[P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho\end{aligned}$$

Conf(M), part of a bigger disconnected group, is the connected component containing the identity transformation in the group of all conformal transformations. Contains all the group properties and they are smooth transformations and the inverses are smooth too. (connected in the open compact connect topology where norm is defined). Integrating these transformations gives the global transformations because the infinitesimal transformations were taken into account first. Since these transformations are continuously connected with the identity of the conformal group, such integrations of the infinitesimal components to get the global transformation can be constructed. That is, the exponential of the generators.Every conformal transformation ϕ that works on any connected subset of the manifold $\mathbb{R}^{p,q}$ is a composition of the aforementioned four kinds of transformations (essentially exponentials of the generators):

- **Translation:** $x^\mu \mapsto x^\mu + c^\mu$ where $c \in \mathbb{R}^{p,q}$.
- **Orthogonal transformation:** $x^\mu \mapsto \Lambda x$ where $\Lambda \in O(p,q)$ (Lorentz transformation in Minkowski spacetime).
- **Dilation:** $x^\mu \mapsto \lambda x^\mu$ where $\lambda \in \mathbb{R}^+$.
- **Special conformal transformation:**

$$x^\mu \mapsto \frac{x^\mu - bx^2}{1 - 2b\cdot x + b^2x^2}, \quad b \in \mathbb{R}^{p,q}.$$

Here, the conformal group is $SO(d+1,1)$ (Euclidean) or $SO(d,2)$ (Minkowski) with $\frac{(d+1)(d+2)}{2}$ number of generators.

2.2. For $d = 2$:

In $d = 2$, when $g_{\mu\nu} = \delta_{\mu\nu}$, the equations 1 and 2 become the Cauchy–Riemann equations:$\partial_1 \epsilon_1 = \partial_2 \epsilon_2, \quad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1$. Since it satisfies the Cauchy–Riemann equations, that implies there is some function whose real part is ϵ^1 and imaginary part is ϵ^2 and the function is analytic in nature. In complex coordinates, let $z = x^0 + ix^1, \quad \bar{z} = x^0 - ix^1$ which implies $\epsilon(z) = \epsilon^0 + i\epsilon^1, \quad \epsilon(\bar{z}) = \epsilon^0 - i\epsilon^1$
From complex analysis, a function will be analytic (complex differentiable) if and only if it satisfies the Cauchy–Riemann equations. Therefore, infinitesimal conformal transformations in $d = 2$ would be of complex analytic functions of nature.

This further implies that in $d = 2$ the conformal factors are not necessarily only quadratic in nature, but could be of any degree of any analytic functions. That is, global conformal transformations (after exponentiating) correspond to entire holomorphic functions $z \mapsto f(z)$, with holomorphic inverse $f^{-1}(z)$. However, Picard's little theorem from complex analysis states that such a function must be linear: $f(z) = \alpha z + \beta, \quad \alpha, \beta \in \mathbb{C}$. This implies that in $d = 2$ locally, conformal transformations could be any holomorphic function, but globally the conformal transformations boil down to only linear holomorphic functions (like Lorentz transformations). Therefore this physically implies:

- Local scale symmetry is larger ⇒ less constraint.
- Global scale symmetry is smaller ⇒ high constraint on the conformal factor.

That is, the local conformal group is much bigger than the global conformal group. Many local conformal generators cannot be exponentiated to form global conformal generators. But this conformal group could be extended a bit more if the infinity is considered within this group, that is, compacting the infinite $\mathbb{R}^{2,0} \to \mathbb{C}$ space: $\mathbb{C} \cup \{\infty\}$— the compactification of $\mathbb{C}$ using the Riemann sphere — which yields:

$$\mathrm{Conf}(\mathbb{C} \cup \{\infty\}) = \left\{ f(z) \;\middle|\; f(z) = \frac{\alpha z + \beta}{\gamma z + \delta},\ \alpha, \beta, \gamma, \delta \in \mathbb{C},\ \alpha\delta - \beta\gamma \neq 0 \right\}$$

This is the group of Möbius transformations. This implies: including infinity with the complex plane enlarges the global conformal group slightly.

For dimension $d = 2$, local conformal transformation group corresponds to any holomorphic or anti-holomorphic function.(By putting $d = 2$ in equations 1 & 2).

$$z \to f(z) \quad \text{(global holomorphic conf. trans.)}$$

$$\bar{z} \to \bar{f}(\bar{z}) \quad \text{(global antiholomorphic trans.)}$$

For infinitesimal case, $z \to z + \epsilon(z), \qquad \bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$
Expanding Laurent series for $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ (around 0) implies:

$$\epsilon(z) = \sum_{n=-\infty}^{\infty} \epsilon_n z^{n+1}, \qquad \bar{\epsilon}(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{\epsilon}_n \bar{z}^{n+1}$$

In general,

$$g(z + \epsilon) = g(z) + \epsilon \frac{\partial g}{\partial z} + \mathcal{O}(\epsilon^2), \qquad \bar{g}(\bar{z} + \bar{\epsilon}) = \bar{g}(\bar{z}) + \bar{\epsilon} \frac{\partial \bar{g}}{\partial \bar{z}} + \mathcal{O}(\bar{\epsilon}^2)$$

That is, $\epsilon \frac{\partial}{\partial z}$ and $\bar{\epsilon} \frac{\partial}{\partial \bar{z}}$ are the required vector fields that generate the transformations:
$\delta g = \epsilon \frac{\partial g}{\partial z}, \qquad \delta \bar{g} = \bar{\epsilon} \frac{\partial \bar{g}}{\partial \bar{z}}$

Let $V = \sum \epsilon_n z^{n+1} \partial_z, \quad n \in \mathbb{Z}$. The corresponding generator is $\ell_n = z^{n+1} \partial_z$.

Similarly, for the anti-holomorphic part, $V = \sum \bar{\epsilon}_n \bar{z}^{n+1} \partial_{\bar{z}}, \quad n \in \mathbb{Z}$, with generators $\bar{\ell}_n = \bar{z}^{n+1} \partial_{\bar{z}}$. Therefore, the generators for $d = 2$ are:

$$\ell_n = z^{n+1} \partial_z, \qquad \bar{\ell}_n = \bar{z}^{n+1} \partial_{\bar{z}},$$

which are differential operators. Here, the Lie product is induced by the Lie bracket corresponding to the commutator of the generators.

For the holomorphic generators:

$$\begin{aligned}[\ell_n, \ell_m] &= \ell_n(\ell_m) - \ell_m(\ell_n) \\ &= z^{n+1}\partial_z\big(z^{m+1}\partial_z\big) - z^{m+1}\partial_z\big(z^{n+1}\partial_z\big) \\ &= z^{n+1}(m+1)z^m\partial_z - z^{m+1}(n+1)z^n\partial_z \\ &= (m-n)z^{m+n+1}\partial_z \\ &= (m-n)\ell_{m+n}.\end{aligned}$$

Similarly, for the anti-holomorphic generators:

$$\begin{aligned}[\bar{\ell}_n, \bar{\ell}_m] &= \bar{z}^{n+1}\partial_{\bar{z}}\big(\bar{z}^{m+1}\partial_{\bar{z}}\big) - \bar{z}^{m+1}\partial_{\bar{z}}\big(\bar{z}^{n+1}\partial_{\bar{z}}\big) \\ &= \bar{z}^{n+1}(m+1)\bar{z}^m\partial_{\bar{z}} - \bar{z}^{m+1}(n+1)\bar{z}^n\partial_{\bar{z}} \\ &= (m-n)\bar{z}^{m+n+1}\partial_{\bar{z}} \\ &= (m-n)\bar{\ell}_{m+n}.\end{aligned}$$

This algebra is called Witt algebra and the central extension of this algebra in quantum field theory is the Virasoro algebra with central charge. The Virasoro algebra is basically the projective representation of this conformal group.

The algebra can be extended to the compact Riemann sphere by including the infinity within $\mathbb{C}$. Essentially, this action extends the global conformal group for $d = 2$ from only linear holomorphic or linear antiholomorphic complex functions to the Mobius group. In this case, conditions are imposed on complex coordinate z and $\bar{z}$ so that they are defined at infinity and finally integrating only the generators which are defined at infinity yields the global conformal group for the compact Riemann sphere.
On the compact Riemann sphere, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$:
On the z–chart a holomorphic conformal vector field is $v_n(z) = z^{n+1}\partial_z$,
Change to $w = \frac{1}{z}$: $\quad \partial_z = -w^2\partial_w$,
$v_n = z^{n+1}\partial_z = w^{-n-1}(-w^2\partial_w) = -w^{-n+1}\partial_w$,
Holomorphic at $w = 0 \iff -w^{-n+1}$ has no pole $\Rightarrow -n+1 \geq 0 \Rightarrow n \leq 1$,
Holomorphic at $z = 0 \iff z^{n+1}$ has no pole $\Rightarrow n+1 \geq 0 \Rightarrow n \geq -1$,
Therefore $n \in \{-1, 0, 1\}$

Now when $n = -1$, $\ell_n = \partial_z \quad \bar{\ell}_n = \partial_{\bar{z}}$
when $n = 0$, $\ell_n = z\partial_z \quad \bar{\ell}_n = \bar{z}\partial_{\bar{z}}$
when $n = 1$, $\ell_n = z^2\partial_z \quad \bar{\ell}_n = \bar{z}^2\partial_{\bar{z}}$
An infinitesimal transformation is, (for holomorphic) $z \longrightarrow z + \epsilon v(z)$ where

$$v(z) = a\ell_{-1} + b\ell_0 + c\ell_1$$

$= a\partial_z + bz\partial_z + cz^2\partial_z$
$= (a + bz + cz^2)\partial_z$
That is, $\delta z = \epsilon(a + bz + cz^2)$, similarly, $\delta\bar{z} = \bar{\epsilon}(\bar{a} + \bar{b}\bar{z} + \bar{c}\bar{z}^2)$.

Now for the global transformation solving the above differential equation $\dfrac{dz}{d\epsilon} = a + bz + cz^2 \quad$ (let $z(0) = c'$) $(a, b, c, c' \in \mathbb{C})$

- $\dfrac{dz}{d\epsilon} = a \to \boxed{z = a\epsilon + c'}$ (translation)
- $\dfrac{dz}{d\epsilon} = bz \to \boxed{z = c'e^{b\epsilon}}$ (rotation+ dilation)
- $\dfrac{dz}{d\epsilon} = cz^2 \to \boxed{z = \dfrac{c'}{1 - c'\epsilon c}}$ (special ct)

This implies, $z(\epsilon) = a\epsilon + c'e^{b\epsilon} + \dfrac{c'}{1 - c'\epsilon c}$ and global conformal group for the $2d$ compact Riemann sphere is $\frac{SL(2,\mathbb{C})}{\mathbb{Z}_2}$.

3. Consequences of Conformal Symmetry on Energy Momentum Tensor of the Field in $d = 2$

Complexification of fields: In terms of complex coordinates, $z = x^0 + ix^1 \qquad \bar{z} = x^0 - ix^1$

Complexification of real coordinates implies mapping $\mathbb{R}^2 \to \mathbb{C}^2$ means $\phi(x^0, x^1) \to \phi(z, \bar{z})$. $\{x^0, x^1\} \in \mathbb{R}$ and $\{z, \bar{z}\} \in \mathbb{C}^2$.

Definition: Fields only depending on z that is $\phi(z)$ are called chiral fields and fields only depending on $\bar{z}$ is $\phi(\bar{z})$ are called anti-chiral fields (holomorphic, anti-holomorphic)

Definition: If $\phi(z, \bar{z}) \to \phi'(z, \bar{z}) = \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda} \bar{z})$ where $z \to \lambda z$ then conformal dimensions are $(h, \bar{h})$. Again if $z \to f(z)$ then $\phi(z, \bar{z}) \to \tilde{\phi}(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial f}{\partial z}\right)^{\bar{h}}$ and $f \in \frac{SL(2,\mathbb{C})}{\mathbb{Z}_2}$ (global transformations) then ϕ is called a quasi-primary field. (primary field $\Rightarrow$ quasi primary field)

Infinitesimal Conformal Transformation of Primary Fields: Let primary field: $\phi(z, \bar{z})$ Probing its behaviour under infinitesimal conformal transformations: $f(z) = z + \epsilon(z)$ where $\epsilon(z) \ll 1$

Now, $\frac{\partial f}{\partial z}) = 1 + h\partial_z \epsilon(z) + O(\epsilon^2)$ (simply binomial & taylor series expansion)
$\phi(z + \epsilon(z), \bar{z}) = \phi(z) + \epsilon(z)\partial_z \phi(z, \bar{z}) + O(\epsilon^2)$
This implies →

$$\phi(z, \bar{z}) \longrightarrow \phi(z, \bar{z}) + (h\partial_z \epsilon + \epsilon \partial_z + \bar{h}\partial_{\bar{z}} \bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}})\phi(z, \bar{z})$$

$$\therefore \delta\phi(z, \bar{z})\Big|_{\epsilon, \bar{\epsilon}} = (h\partial_z \epsilon + \epsilon \partial_z + \bar{h}\partial_{\bar{z}} \bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}})\phi(z, \bar{z})$$

Energy momentum tensor can be deduced from the variation of the action with respect to the metric. This implies, the energy momentum tensor encapsulates the infinitesimal transformations of metric.

Implication of conformal invariance on energy momentum tensor: Noether's theorem: Every continuous symmetry in a field theory → existence of a current j_μ which is conserved. That is,

$$\partial^\mu j_\mu = 0$$

for the infinitesimal conformal transformation, $x^\mu \longrightarrow x^\mu + \epsilon^\mu(x)$ implies $\exists$ a conserved current such as, $j_\mu = T_{\mu\nu}\, \epsilon^\nu$symmetric in nature called energy momentum tensor.

Now, $\partial^\mu j_\mu = 0 \to \partial^\mu (T_{\mu\nu}\epsilon^\nu) = 0 \Rightarrow \quad \partial^\mu T_{\mu\nu} = 0 \quad$ as $\epsilon^\nu =$ constant
Again when ϵ^ν depends on coordinates,

$$\begin{aligned}
& \partial^\mu j_\mu = 0 \\
\Rightarrow \quad & \partial^\mu (T_{\mu\nu}\epsilon^\nu) = 0 \\
\Rightarrow \quad & T_{\mu\nu}\partial^\mu \epsilon^\nu + \partial^\mu T_{\mu\nu}\, \epsilon^\nu = 0 \\
\Rightarrow \quad & T_{\mu\nu}\partial^\mu \epsilon^\nu = 0 \\
\Rightarrow \quad & T_{\mu\nu} \cdot \tfrac{1}{2} \cdot \{\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu\} = 0 \\
\Rightarrow \quad & T_{\mu\nu} \tfrac{1}{2} \mathrm{tr}\, (\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu) = 0 \\
\Rightarrow \quad & T_{\mu\nu} \tfrac{1}{2} \cdot \tfrac{2}{d}(\partial \cdot \epsilon) g^{\mu\nu} = 0 \quad \text{(from equation 1)} \\
\Rightarrow \quad & \tfrac{1}{d} T^\mu{}_\mu (\partial \cdot \epsilon) = 0 \qquad [\text{on } \mathbb{R}^d \text{ euclidean}]
\end{aligned}$$

implies, in CFT, the energy momentum tensor $T_{\mu\nu}$ is traceless.

For d = 2, (euclidean) consequences of traceless energy momentum tensor on 2D qft for euclidean space time:

$T_{\mu\nu} = \frac{\partial x^\sigma}{\partial x^\mu} \frac{\partial x^\rho}{\partial x^\nu} T_{\sigma\rho}$ for $x^0 = \frac{1}{2}(z + \bar{z})$ $x^1 = \frac{1}{2i}(z - \bar{z})$

Converting the energy momentum tensor in complex coordinates via the above coordinate transformation, it is obtained that T_{zz} and $T_{\bar{z}\bar{z}}$ are two non vanishing components of the tensor where $T_{zz}(z, \bar{z}) \; =: \; T(z)$ and $T_{\bar{z}\bar{z}}(z, \bar{z}) \; =: \; \bar{T}(\bar{z})$ which physically implies that for any conformal symmetric field theory in Eucliden 2 dimensions, the energy momentum tensor is traceless and can be decomposed to chiral and anti-chiral parts. These chiral and anti-chiral field components generate the separate holomorphic and anti-holomorphic Witt algebras.

4. An example: $2d$ Classical Free Scalar Field

The free scalar field : example of classical scalar field : For the free scalar field action :

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial_\alpha X \partial^\alpha X$$

this is a free scalar field action $\implies$ no interaction terms and the action is quadratic. Here σ^α where $\alpha = 0, 1$ and $\alpha = 0$ is usually time (proper) and $\alpha = 1$ is the σ space coordinate in the 2D euclidean sheet. Metric has all positive signs. Also a bit of notation : $d^2\sigma \equiv d\sigma^0 d\sigma^1$. Here $X(\sigma)$ is the scalar field. $X : (\sigma^0, \sigma^1)$ $\mapsto$ position of scalar particle in spacetime at each σ^α.

Conformal invariance : A 2D theory is conformal if the action is invariant under dilations. Let $\sigma^\alpha \longrightarrow \lambda\sigma^\alpha$ $X(\sigma) \longrightarrow X(\lambda^{-1}\sigma)$ and $\frac{\partial X(\sigma)}{\partial \sigma^\alpha} \longrightarrow \frac{\partial X(\lambda^{-1}\sigma)}{\partial \sigma^\alpha} = \frac{1}{\lambda}\frac{\partial X(\lambda^{-1}\sigma)}{\partial(\lambda^{-1}\sigma)}$

The action becomes :

$$S' = \frac{1}{4\pi\alpha'} \int \lambda^2 \frac{1}{\lambda^2} \partial_\alpha X \partial^\alpha X = S$$

The action remains invariant under dilation. In terms of algebraic calculation, checking the dilation transformation is easier, however, transforming the coordinates in terms of elements of the Poincare(Lorentz and translation) group and Special conformal transformations is non-trivial. Therefore, based on calculations from section 3, if the energy momentum tensor for this field is traceless along with can be decomposed into chiral and anti-chiral components that would imply $2d$ classical free scalar field is conformal symmetric field.
$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \partial_\alpha X \, \partial^\alpha X$
Energy momentum tensor, $T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g^{\alpha\beta}}$ and for the euclidean space $g_{\alpha\beta} = \delta_{\alpha\beta} = \delta^{\alpha\beta} = g^{\alpha\beta}$ and $\sqrt{g} = 1$
$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{g} \, g^{\alpha\beta} \, \partial_\alpha X \, \partial_\beta X$
$\delta S = \delta \left\{ \frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{g} g^{\alpha\beta} \, \partial_\alpha X \, \partial_\beta X \right\}$
$= \frac{1}{4\pi\alpha'} \int d^2\sigma \, \delta \left(\sqrt{g} \, g^{\alpha\beta} \right) \partial_\alpha X \, \partial_\beta X$
$= \frac{1}{4\pi\alpha'} \int d^2\sigma \, \delta\sqrt{g} \, g^{\alpha\beta} \, \partial_\alpha X \, \partial_\beta X + \frac{1}{4\pi\alpha'} \int d^2\sigma \, \sqrt{g} \, \delta g^{\alpha\beta} \, \partial_\alpha X \, \partial_\beta X$

$\delta S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\delta\sqrt{g} g^{\alpha\beta} + \sqrt{g} \delta g^{\alpha\beta} \right) d_\alpha X d_\beta X$
$[\delta\sqrt{g} = -\frac{\sqrt{g} g_{\rho\sigma} \delta g^{\rho\sigma}}{2}$; deduced from $g^{-1}g = I, \ln\det A = \text{Tr} \ln A]$
$= \frac{1}{4\pi\alpha'} \int d^2\sigma \left(-\frac{\sqrt{g}}{2} g \delta g g^{\alpha\beta} + \sqrt{g} \delta g^{\alpha\beta} \right) d_\alpha X d_\beta X$
$= \frac{1}{4\pi\alpha'} \int d^2\sigma \left(-\frac{\sqrt{g}}{2} g_{\rho\sigma} \delta g^{\rho\sigma} g^{\alpha\beta} + \sqrt{g} \delta g^{\alpha\beta} \right) d_\alpha X d_\beta X$
$= \frac{\sqrt{g}}{4\pi\alpha'} \int d^2\sigma \left(-\frac{g_{\rho\sigma}}{2} \delta g^{\rho\sigma} g^{\alpha\beta} + \delta g^{\alpha\beta} \right) d_\alpha X d_\beta X$
$= \frac{\sqrt{g}}{4\pi\alpha'} \int d^2\sigma \left\{ -\frac{g_{\rho\sigma}}{2} \delta g^{\rho\sigma} g^{\alpha\beta} d_\rho X d_\sigma X + \delta g^{\alpha\beta} d_\alpha X d_\beta X \right\}$
$= \frac{\sqrt{g}}{4\pi\alpha'} \int d^2\sigma \left\{ -\frac{g_{\alpha\beta}}{2} \delta g^{\alpha\beta} g^{\rho\sigma} d_\rho X d_\sigma X + \delta g^{\alpha\beta} d_\alpha X d_\beta X \right\}$
$= \frac{\sqrt{g} \delta g^{\alpha\beta}}{4\pi\alpha'} \int d^2\sigma \left\{ -\frac{g_{\alpha\beta}}{2} d_\rho X d^\rho X + d_\alpha X d_\beta X \right\}$
$= \frac{\sqrt{g} \delta g^{\alpha\beta}}{4\pi\alpha'} \int d^2\sigma \left\{ -\frac{g_{\alpha\beta}}{2} (dX)^2 + d_\alpha X d_\beta X \right\}$
Now $T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g^{\alpha\beta}}$
$= -\frac{4\pi}{\sqrt{g}} \frac{1}{\delta g^{\alpha\beta}} \frac{\sqrt{g} \delta g^{\alpha\beta}}{4\pi\alpha'} \left\{ -\frac{g_{\alpha\beta}}{2} (dX)^2 + d_\alpha X d_\beta X \right\}$
$= -\frac{1}{\alpha'} \left\{ d_\alpha X d_\beta X - \frac{1}{2} \delta_{\alpha\beta} (dX)^2 \right\} \qquad (g_{\alpha\beta} = \delta_{\alpha\beta}, \text{ Euclidean})$
That is , $T_{\alpha\beta} = -\frac{1}{\alpha'} \left(d_\alpha X d_\beta X - \frac{1}{2} \delta_{\alpha\beta} (dX)^2 \right)$
$T^\alpha_{\ \alpha} = g^{\alpha\mu} T_{\mu\alpha} = \delta^{\alpha\mu} \left[-\frac{1}{\alpha'} \left(d_\mu X d_\alpha X - \frac{1}{2} \delta_{\mu\alpha} (dX)^2 \right) \right] = -\frac{1}{\alpha'} \left\{ d^\alpha X d_\alpha X - (dX)^2 \right\} = 0$
This further implies, the energy momentum tensor is traceless and in complex coordinates:
$z = \sigma^0 + i\sigma^1 \qquad \bar{z} = \sigma^0 - i\sigma^1$ (from section 3) the energy momentum tensor for this $2d$ free scalar field has chiral and anti-chiral components which generate seperate Witt algebras.

5. Motivation towards Quantum Conformal Field theory and Conclusion:

So far various calculations and implications of conformal symmetries of classical fields have been explained in this report. The previous section raises the question, 'What kind of constraints do quantum conformal field theories imply?' Quantum CFT is in itself a field to explore for its mathematical richness in exploring different Lie algebras, including its physical implications in understanding different quantum statistical phenomena like phase transitions, Ising models, etc.(Blumenhagen, 2009, p.1-2). Next, explicitly quantum conformal field theories can be explored.

In Section 2, it has been mentioned that the Witt algebra can be extended to form the Virasoro algebra, which is the $2d$ quantum conformal algebra. In terms of group representation, the Witt algebra can be extended using the projective representation of the group, giving rise to a central charge in the Virasoro algebra. Similarly, quantum conformal symmetry group generators can be identified, and their corresponding Lie algebras can also be determined, which is the required conformal algebra. In the classical conformal group, the generators of the symmetry are vector fields which act on the underlying manifold's coordinates, whereas after quantising the classical fields, the corresponding generators for the quantum conformal symmetry group convert into operators which act on the states of the Hilbert space. Therefore, a different mechanism needs to be looked into for what it even means to multiply operators on the Hilbert space, which is essential here to compute the corresponding Lie brackets between different generators for the quantum conformal symmetry group. This is where the operator product expansion comes in, which is already an integral part of QFTs, but this can be explored in the context of QCFT. Furthermore, for QFTs, it can be shown that the Virasoro algebra with central charges emerges independently of the Witt algebra, making the two algebras effectively equivalent. Finally, through radial quantisation of the fields, where explicitly the fields are now quantised into local operators (depending on spacetime coordinates), the energy-momentum tensor is converted into a local operator. However, one subtlety here is that by radially quantising classical fields into quantum fields, for some fields, it results in a non-zero trace of the operator energy-momentum tensor, whereas in classical conformal fields, the energy-momentum tensors are always traceless. This anomaly is known as the Weyl anomaly, which can be explored further.

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7. References

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Approval

The internship report titled “ Conformal Field Thoery” submitted by Fabiha Noshin, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfillment of the requirements of the internship program. The internship was conducted under the supervision of **Samanta Saha** during the period **15 July 2025** to **15 October 2025**.

Supervisor

Samanta Saha

Samanta Saha

Samanta Saha
Case Western Reserve University