Understanding the Hodge Decomposition and Elliptic Regularity Theorem

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1 Introduction

The theory of elliptic partial differential equations (PDEs) lies at the foundation of modern analysis and differential geometry. A central objective in this theory is to understand how the analytic properties of solutions reflect the underlying geometry of the manifold. In this project, we focus on two fundamental results in this direction: the Hodge decomposition theorem and the elliptic regularity.

The interior elliptic estimate provides quantitative control of the derivatives of a solution in terms of the data of the equation. It ensures that weak solutions to elliptic equations are, in fact, smooth under appropriate conditions, a key step toward establishing elliptic regularity. By completing the proof of this theorem, we confirm the crucial analytic mechanism that guarantees the regularity of solutions inside the domain, independent of boundary behavior.

Building on this analytic foundation, the Hodge decomposition theorem offers a powerful geometric interpretation. It asserts that every differential form on a compact oriented Riemannian manifold can be uniquely expressed as the sum of an exact, a co-exact, and a harmonic form. This decomposition links the analytic structure defined by the Laplace–Beltrami operator to the topological structure captured by de Rham Cohomology.

Together, these results illustrate the deep interplay between analysis and geometry: elliptic estimates ensure the smoothness and compactness properties required for the Hodge theory, while the Hodge decomposition reveals how analytic methods recover topological invariants. In this project, we studied the proof of the Hodge decomposition from [War71] and attempted to present the necessary concepts, together with proper motivation, that require reading [Bre10] and [Eva10].

2 Hodge Theorem

We follow the notation of [War71]. In this section, our main goal is to demystify Theorem 6.8 of [War71].

Theorem 2.1. *Let $\alpha \in E^p(M)$, and let l be a weak solution of $\Delta\omega = \alpha$. Then there exists $\omega \in E^p(M)$ such that*

$$l(\beta) = \langle \omega, \beta \rangle$$

for every $\beta \in E^p(M)$. Consequently, $\Delta\omega = \alpha$.

Observation: This theorem bridges a key gap between weak solutions and classical (smooth) solutions. In many differential equations, we first look for weak solutions because they always exist under mild assumptions. But such solutions aren't necessarily smooth. What this theorem tells us is that something special happens for the Laplacian. Every weak solution is actually a smooth solution. This is a deep property of elliptic operators like the Laplacian. That is, even if ω was only defined via a weak formulation, the structure of Δ guarantees the existence of a smooth representative.

Theorem 2.2. *Let $\{\alpha_n\}$ be a sequence of smooth p-forms on M such that $\|\alpha_n\| \leq c$ and $\|\Delta\alpha_n\| \leq c$ for all n and for some constant $c > 0$. Then a subsequence of $\{\alpha_n\}$ is a Cauchy sequence in $E^p(M)$.*

Lemma 2.1. *There is a constant $c > 0$ such that for all $\beta \in H^p(M)^\perp$ we have*

$$\|\beta\| \leq c\|\Delta\beta\|$$

Proof. Suppose we doesn't have such bound. Then there exist a sequence $\alpha_n \in H^p(M)^\perp$ such that,

$$\|\alpha_n\| > n\|\Delta\alpha_n\|$$

for all n. We can set, $\beta_n = \frac{1}{\|\alpha_n\|}\alpha_n$. So, now we have orthonormal sequence, $\|\beta_n\| = 1$. And, $\|\Delta\beta_n\| = \frac{1}{\|\alpha_n\|}\|\Delta\alpha_n\| \leq \frac{1}{n}$. By theorem 2.2 (how does we show the boundeness), a subsequence of the β_n, which for convenience we can assume to be $\{\beta_n\}$ itself, is Cauchy. Thus $\lim_{j\to\infty}\langle\beta_j, \psi\rangle$ exists for each $\psi \in E^p(M)$. We define a linear functional l on $E^p(M)$ by setting

$$l(\psi) = \lim_{j\to\infty}\langle\beta_j, \psi\rangle, \ \forall\psi \in E^p(M)$$

And,

$$|l(\psi)| = \left| \lim_{j\to\infty} \langle \beta_j, \psi \rangle \right| \leq \lim_{j\to\infty} \|\beta_j\| \|\psi\| = \|\psi\|$$

Which means l is bounded.

$$l(\Delta\varphi) = \lim_{j\to\infty} \langle \beta_j, \Delta\varphi \rangle = \lim_{j\to\infty} \langle \Delta\beta_j, \varphi \rangle = 0$$

So, l is a weak solution of $\Delta\beta = 0$. By theorem 2.2, there exists $\beta \in E^p(M)$ such that $l(\psi) = \langle \beta, \psi \rangle$.

Actually, we construct this functional to leverage the completeness of $\mathbb{R}$. Because,

$$|\langle \beta_i, \psi \rangle - \langle \beta_j, \psi \rangle| = |\langle \beta_i - \beta_j, \psi \rangle| \leq \|\beta_i - \beta_j\| \|\psi\| < \epsilon$$

This sequence is Cauchy using the fact that our $\{\beta_j\}$ is Cauchy, which means $\beta_j \to \beta$. So, $\{\langle \beta_j, \psi \rangle\}$ converges to some number which belong to $\mathbb{R}$. Okay, then

$$0 = l(\psi) - l(\psi) = \lim_{j\to\infty} \langle \beta_j, \psi \rangle - \langle \beta, \psi \rangle = \langle \lim_{j\to\infty} \beta_j - \beta, \psi \rangle$$

Here, $\beta_j - \beta$ converges to $\beta' - \beta$, which gives $\langle \beta' - \beta, \psi \rangle = 0$ for all ψ. but that can be possible only when $\beta' - \beta = 0$ as $\langle -, - \rangle$ is non-degenerate. Hence, we are now settled,

$$l(\psi) = \lim_{j\to\infty} \langle \beta_j, \psi \rangle = \langle \beta, \psi \rangle, \quad \text{where } \beta_j \to \beta$$

Since, $\|\beta_j\| = 1$ and $\|\beta_j\| \in (H^p(M))^\perp$.

$$\begin{aligned}
\|\beta\|^2 &= l(\beta) \\
&= \lim_{j\to\infty} \langle \beta_j, \beta \rangle \\
&= \lim_{j\to\infty} \lim_{k\to\infty} \langle \beta_j, \beta_k \rangle \\
&= \lim_{j\to\infty} \lim_{k\to\infty} \langle \beta_j - \beta_k + \beta_k, \beta_k \rangle \\
&= \lim_{j\to\infty} \left(\lim_{k\to\infty} \langle \beta_j - \beta_k, \beta_k \rangle + (\beta_k, \beta_k \rangle \right) \\
&= 1 + \lim_{j\to\infty} \left(\lim_{k\to\infty} \langle \beta_j - \beta_k, \beta_k \rangle \right)
\end{aligned}$$

But,

$$|\langle \beta_j - \beta_k, \beta_k \rangle| \leq \|\beta_j - \beta_k\| \|\beta_k\| = \|\beta_j - \beta_k\| \leq \epsilon$$

it follows that $\|\beta\| = 1$ and $\beta_j \in H^p(M)^\perp$. The complement of $H^p(M)$ is a closed subspace of the Hilbert space while completing $E^p(M)$. Thus, $\beta \in H^p(M)^\perp$. But by theorem 2.2, $\Delta\beta = 0$, so $\beta \in H^p(M)$. That means,

$$\beta \in H^p(M) \cap H^p(M)^\perp \implies \beta = 0$$

which is a contradiction. Thus, the lemma is proved. $\square$

Theorem 2.3. ***(The Hodge Decomposition Theorem):*** *For each integer p with $0 \leq p \leq n$, $H^p(M)$ is finite dimensional, and we have the following orthogonal direct sum decompositions of the space $E^p(M)$ of smooth p-forms on M :*

$$\begin{aligned} E^p(M) &= \Delta\left(E^p(M)\right) \oplus H^p(M) \\ &= d\delta\left(E^p\right) \oplus \delta d\left(E^p\right) \oplus H^p \\ &= d\left(E^{p-1}\right) \oplus \delta\left(E^{p+1}\right) \oplus H^p \end{aligned} \tag{2.1}$$

Consequently, the equation $\Delta\omega = \alpha$ has a solution $\omega \in E^p(M)$ if and only if the p-form α is orthogonal to the space of harmonic p-forms.

Motivation: Before we start writing the proof, let's ask why this theorem is important. There are two important observations. The equation $\Delta\omega = 0$ defines harmonic forms. Surprisingly, this solution space is finite-dimensional on a compact manifold. This contrasts with simpler differential equations like

$$\frac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad f = f(x),$$

whose solution space is infinite-dimensional. The finiteness reflects deep geometric structure and is not typical for general differential operators. Another is, in linear algebra, we solve $Ax = b$ by projecting b onto the image of A. Similarly, Hodge decomposition expresses any form α as $\alpha = d\beta + \delta\gamma + h$, where h is harmonic. This lets us solve equations like $\Delta\omega = \alpha$ by inverting Δ on the orthogonal complement of its kernel — a geometric analog of solving nonhomogeneous linear systems.

Proof. If $H^p(M)$ were not finite-dimensional, then $H^p(M)$ would contain an infinite orthonormal sequence. Then each element $\|\omega_i\| = 1$ and $\|\Delta\omega_i\| = 0 < 1$. Hence, considering $c = 1$, we can apply theorem 2.1. Which means this orthonormal sequence would contain a Cauchy subsequence, which is impossible because,

$$\begin{aligned} \|\omega_i - \omega_j\|^2 &= \|\omega_i\| + \|\omega_j\| - 2\langle\omega_i, \omega_j\rangle \\ &= 1 + 1 - 2 \cdot 0 \\ &= 2 \end{aligned}$$

Thus, $H^p(M)$ is finite-dimensional. We are left to see

$$E^p(M) = \Delta(E^p) \oplus H^p(M)$$

Let $\omega_1, \cdots, \omega_l$ be an orthogonal basis of $H^p(M)$. Let H denote the projection operator and for any $\alpha \in E^p(M)$ set,

$$H(\alpha) = \sum_{i=1}^{l} \langle\alpha, \omega_i\rangle\omega_i$$

Then for every $\alpha \in E^p(M)$ we can write,

$$\alpha = H(\alpha) + (\alpha - H(\alpha))$$

where $\alpha - H(\alpha) \in H^p(M)^\perp$ because

$$\begin{aligned}
\langle \alpha - H(\alpha), \omega_i \rangle &= \left\langle \alpha - \sum_{i=1}^{l} \langle \alpha, \omega_i \rangle \omega_i, \omega_i \right\rangle \\
&= \langle \alpha, \omega_i \rangle - 0 - \cdots - \langle \langle \alpha, \omega_i \rangle \omega_i, \omega_i \rangle - 0 \cdots \\
&= \langle \alpha, \omega_i \rangle - \langle \alpha, \omega_i \rangle \\
&= 0
\end{aligned}$$

Thus, we have an orthogonal direct sum decomposition,

$$E^p(M) = H^p(M) \oplus H^p(M)^\perp$$

So, we are left to show that $H^p(M)^\perp = \Delta(E^p(M))$. One inclusion is easy to show. Suppose, $\omega \in E^p(M)$ so that there is some $\tilde{\omega}$ such that $\omega = \Delta\tilde{\omega}$. Thus, for all $\alpha \in H^p(M)$ we have,

$$\begin{aligned}
\langle \omega, \alpha \rangle &= \langle \Delta\tilde{\omega}, \alpha \rangle \\
&= \langle \tilde{\omega}, \Delta\alpha \rangle \\
&= \langle \tilde{\omega}, 0 \rangle \\
&= 0
\end{aligned}$$

Hence,

$$\Delta(E^p(M)) \subset H^p(M)^\perp \tag{2.2}$$

To complete the proof, we need to show that for any $\alpha \in H^p(M)^\perp$ there is some ω such that $\Delta\omega = \alpha$. Okay, doesn't it seem to find a weak solution $l \in E^p(M)^*$? But instead of getting the weak solution from $E^p(M)^*$, we will target a smaller space $\Delta(E^p(M))$. We begin to construct our weak solution for a given $\alpha \in H^p(M)^\perp$ by defining a linear functional $l : \Delta(E^p(M)) \to \mathbb{R}$ by setting,

$$l(\Delta\varphi) = \langle \alpha, \varphi \rangle, \ \forall \varphi \in E^p(M)$$

Now, l is well-defined since if $\Delta\omega_1 = \Delta\omega_2$, then $\varphi_1 - \varphi_2 \in H^p(M)$, so that $\langle \alpha, \varphi_1 - \varphi_2 \rangle = 0$. For any $\varphi \in E^p(M)$ and let $\psi = \varphi - H(\varphi)$. Then $\Delta\psi = \Delta\varphi - \Delta H(\varphi) = \Delta\varphi$. Using lemma 2.1 we get,

$$\begin{aligned}
\|l(\Delta\varphi)\| &= \|l(\Delta\psi)\| \\
&= \|\langle \alpha, \psi \rangle\| \\
&\overset{\text{CS}}{\leq} \|\alpha\| \ \|\psi\| \\
&\overset{2.1}{\leq} c\|\alpha\| \ \|\Delta\psi\| \\
&= c\|\alpha\| \ \|\Delta\varphi\| \\
&= K\|\Delta\varphi\|
\end{aligned}$$

So, there exists $K > 0$ such that $\|l(\Delta\varphi)\| \leq K\|\Delta\varphi\|$ for all $\varphi \in \Delta(E^p(M))$. Hence, l is a bounded linear functional. Then we can apply the Hahn-Banach theorem to our linear functional to extend l to all of $E^p(M)$. What does it mean?

$$l : E^p(M) \xrightarrow{?} \mathbb{R} \tag{2.3}$$

$$\begin{aligned}
l(\Delta\varphi) &= \langle \alpha, \varphi \rangle \\
\underset{\exists!\omega}{\overset{\text{Riesz}}{\Longrightarrow}} \langle \omega, \Delta\varphi \rangle &= \langle \alpha, \varphi \rangle \\
\overset{\text{self-adjoint}}{\Longrightarrow} \langle \Delta\omega, \varphi \rangle &= \langle \alpha, \varphi \rangle \\
\Delta\omega &= \alpha
\end{aligned}$$

Thus, l is a weak solution of $\Delta\omega = \alpha$. We can again apply Theorem 2.1. There exists $\omega \in E^p(M)$ such that $\Delta\omega = \alpha$. That means for $\alpha \in H^p(M)^\perp$, we get $\Delta\omega = \alpha$ where $\omega \in E^p(M)$. Hence,

$$H^p(M)^\perp \subset \Delta(E^p(M)) \tag{2.4}$$

From 2.2 and 2.4, we get $H^p(M)^\perp = \Delta(E^p(M))$. And the Hodge decomposition theorem is proved. □

To complete the proof, we must also show that Δ has a closed range, a fact that is clearly presented in [Est18]. Another helpful resource that provides a complete proof of the Hodge theorem for both real and complex manifolds is [Ush16].

3 Sobolev Space

Initially, our goal was to study [Bre10] and [Eva10] and write a complete exposition of Sobolev spaces. Our initial goal was to define the Schwartz space and the Sobolev space on $\mathbb{R}^d$. We also define the topologies of these spaces. And move on to the full exposure of the test function space, $\mathcal{D}(U)$, and its dual space, the tempered distribution $\mathcal{D}'(U)$, where we will use the countable family of semi-norms to generate a smooth topology over $\mathcal{D}(U)$. However, due to time constraints, we will instead discuss a few essential facts and then proceed directly to Theorem 1 in Section 6.3.1 of [Eva10].

We will start our journey from the very primitive object. Let's start with the space of differentiable functions with continuous derivatives C^1 (I said very primitive, not ancient, that's why skip C^0). You may ask why we are so obsessed with C^1 space. Okay, the Sobolev space $W^{k,p}$ bridges the gap between integrable functions and smooth functions. To make everything intuitive, we will discuss only the $W^{1,p}$ space here. Let $I = (a, b)$ be an open interval and let $p \in \mathbb{R}$ with $1 \leq p < \infty$.

Figure 1: Mathematician Haji

Definition 3.1. The Sobolev space $W^{1,p}(I)$ is defined to be,

$$W^{1,p} = \left\{ u \in L^p : \exists g \in L^p \text{ such that } \int_I u g' = -\int_I g\varphi ; \forall \varphi \in C_c^1(I) \right\}$$

Fact: The functions in $C^k(\Omega)$, need not be bounded even if Ω is bounded; for example, $u(x) = \frac{1}{x} \in C^\infty(0,1)$.
Fact: Continuity does not say how rapidly the values $u(y)$ of a function approach its value $u(x)$ as $y \to x$. A way to strengthen the definition of continuity is to require that the modulus of continuity is proportional to a power $|x-y|^\alpha$ for some exponent $0 < \alpha \leq 1$. There is no purpose in considering Hölder continuous function with exponent greater than one, since any such function is differentiable with zero derivative and therefore is constant.
For $0 < \alpha < 1$, define $u(x) = (0,1) \to \mathbb{R}$ by $u(x) = |x|^\alpha$. We know that,

$$||x|^\alpha - |y|^\alpha| \leq |x-y|^\alpha \implies \frac{||x|^\alpha - |y|^\alpha|}{|x-y|^\alpha} \leq 1$$

Hence, $u(x) \in C^{0,\alpha}([0,1])$ but $u \notin C^{0,\beta}([0,1])$ for $\alpha < \beta \leq 1$.

Let $[u]_{\beta,\Omega} = M < \infty$. For any $x, y \in \Omega$ with $0 < |x-y| \leq 1$,

$$|u(x) - u(y)| \leq M|x-y|^\beta \leq M|x-y|^\alpha$$

So, higher exponent regularity implies lower exponent regularity, but the converse is not true.

Remark: In general, there is no relation between the types $L^p \subset L^q$. For example, suppose that $X = (0,1)$ and the function $f(x) = x^{-\frac{1}{2}}$. Then $f \in L^1(0,1)$ but $f \notin L^2(0,1)$. On the other hand, if $X = (1,\infty)$ and $f(x) = x^{-1}$, then $f \in L^2(1,\infty)$, but $f \notin L^1(1,\infty)$.

Theorem 3.1. *(Mollification) Suppose that, $f \in L^p_{loc}(\Omega)$ for $1 \le p < \infty$, and $\epsilon > 0$. Define $f^\epsilon : \Omega^\epsilon \to \mathbb{R}$ where*

$$f^\epsilon(x) = \int_\Omega \eta^\epsilon(x-y)f(y)dy$$

and $\Omega^\epsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}$. then:

1. *$f^\epsilon \in C^\infty(\Omega^\epsilon)$ is smooth;*
2. *$f^\epsilon \to f$ pointwise almost everywhere in Ω as $\epsilon \to 0^+$;*
3. *$f^\epsilon \to f$ in $L^p_{loc}(\Omega)$ as $\epsilon \to 0^+$.*

Fact: To ensure that every ball of radius ϵ centered at x is fully inside Ω, we require: $B_\epsilon(x) \subset \Omega \iff x \in \Omega^\epsilon$. The convolution integral only involves values of f where it is defined, and we have the room to average f.

$\boxed{\text{The space of test functions } C_c^\infty(\Omega) \text{ is dense in } L^p_{loc}(\Omega) \text{ for } 1 \le p < \infty.}$

Fact: A continuous function that is point-wise differentiable almost everywhere need not have a weak derivative. For example: Cantor function.

Theorem 3.2. *Suppose that $f \in L^1_{loc}(\Omega)$ has weak derivative $\partial^\alpha f \in L^1_{loc}(\Omega)$. Then $\eta^\epsilon \star f \in C^\infty(\Omega^\epsilon)$ and*

$$\partial^\alpha(\eta^\epsilon \star f) = \eta^\epsilon \star (\partial^\alpha f)$$

Moreover,

$$\partial^\alpha(\eta^\epsilon \star f) \to \partial^\alpha f, \text{ in } L^p_{loc}(\Omega) \text{as } \epsilon \to 0^+$$

Theorem 3.3. *(Weak derivative as limit of derivatives of smooth functions) A function $f \in L^1_{loc}(\Omega)$ is weakly differentiable in Ω if and only if there is a sequence $\{f_n\}$ of functions $f_n \in C^\infty(\Omega)$ such that $f_n \to f$ and $\partial^\alpha f_n \to g$ in $L^1_{loc}(\Omega)$. In that case, the weak derivative of f is given by $g = \partial^\alpha f \in L^1_{loc}(\Omega)$.*

Theorem 3.4. *For any $k \in \mathbb{N}$ and $1 \le p < \infty$, the space $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.*

4 Derivation of Interior Elliptic Regularity

Okay, in this section, we will present the complete proof of Theorem 1 in Section 6.3.1 of [Eva10].

Theorem 4.1. *Assume $a^{ij} \in C^1(U), b^i, c \in L^\infty(U) \quad (i, j = 1, \cdots, n)$ and $f \in L^2(U)$. Suppose, furthermore, that $u \in H^1(U)$ is a weak solution of the elliptic PDE*

$$Lu = f \quad \text{in } U \tag{4.1}$$

Then $u \in H^2_{loc}(U)$. And for each open subset $V \Subset U$, we have the estimate

$$\|u\|_{H^2(V)} \leq C(\|f\|_L^2(U) + \|u\|_{L^2(U)}), \tag{4.2}$$

the constant C depending only on V, U, and the coefficients of L.

Proof. Fix any open set $V \Subset U$, and choose an open set W such that $V \Subset W \Subset U$. Then select a smooth function ζ satisfying

$$\begin{cases} \zeta \equiv 1 \text{ on } V, \zeta \equiv 0 \text{ on } \mathbb{R}^n - W, \\ 0 \leq \zeta \leq 1 \end{cases}$$

We call ζ a cutoff function. Its purpose in the subsequent calculations will be to restrict all expressions to the subset W, which is a positive distance away from ∂U. This is necessary as we have no information concerning the behavior of u near ∂U. Now, since u is a weak solution of 4.1, we have $B[u, v] = (f, v)$ for all $v \in H^1_0(U)$. Consequently,

$$\sum_{i,j=1}^{n} \int_U a^{ij} u_{x_i} v_{x_j} dx = \int_U \tilde{f} v dx \tag{4.3}$$

where

$$\tilde{f} := f - \sum_{i,j=1}^{n} b^i u_{x_i} - cu \tag{4.4}$$

Now let $|h| > 0$ be small, choose $k \in \{1, \cdots, n\}$. and then substitute

$$v := -D_k^{-h}(\zeta^2 D_k^h u)$$

into 4.3. We write the resulting expression as $A = B$. We write the resulting expression as

$$A = B,$$

for

$$A := \sum_{i,j=1}^{n} \int_U a^{ij} u_{x_i} v_{x_j} dx$$

and

$$B := \int_U \tilde{f} v dx$$

Estimate of A. To estimate A, we need the following two important formulas:

$$\begin{aligned}
\int_U \phi(x) D_k^{-h} \psi(x) dx &= \int_U \phi(x) \frac{\psi(x - he_k) - \psi(x)}{-h} dx \\
&= -\frac{1}{h} \left[\int_U \phi(x)\psi(x - he_k) dx - \int_U \phi(x)\psi(x) dx \right] \\
&= -\frac{1}{h} \left[\int_U \phi(y + he_k)\psi(y) dy - \int_U \phi(x)\psi(x) dx \right] \\
&= -\frac{1}{h} \left[\int_U \phi(x + he_k)\psi(x) dx - \int_U \phi(x)\psi(x) dx \right] \\
&= -\int_U \frac{\phi(x + he_k) - \phi(x)}{h} \psi(x) dx \\
&\overset{(1)}{=} -\int_U D_k^h \phi(x)\psi(x) dx
\end{aligned}$$

$$\begin{aligned}
D_h^k(a^{ij} u_{x_i}) &= \frac{a^{ij}(x + he_k) u_{x_i}(x + he_k) - a^{ij}(x) u_{x_i}(x)}{h} \\
&= \frac{a^{ij}(x + he_k) u_{x_i}(x + he_k) - a^{ij}(x + he_k) u_{x_i}(x)}{h} \\
&\quad + \frac{a^{ij}(x + he_k) u_{x_i}(x) - a^{ij}(x) u_{x_i}(x)}{h} \\
&\overset{(2)}{=} a^{ij,h} D_h^k(u_{x_i}) + D_h^k(a^{ij}) u_{x_i}
\end{aligned}$$

$$\begin{aligned}
A &= -\sum_{i,j=1}^n \int_U a^{ij} u_{x_i} \left[D_k^{-h} \left(\zeta^2 D_k^h u \right) \right]_{x_j} dx \\
&\overset{(1)}{=} \sum_{i,j=1}^n \int_U D_k^h \left(a^{ij} u_{x_i} \right) \left(\zeta^2 D_k^h u \right)_{x_j} dx \\
&\overset{(2)}{=} \sum_{i,j=1}^n \int_U a^{ij,h} \left(D_k^h u_{x_i} \right) \left(\zeta^2 D_k^h u \right)_{x_j} \\
&\quad + \left(D_k^h a^{ij} \right) u_{x_i} \left(\zeta^2 D_k^h u \right)_{x_j} dx
\end{aligned} \tag{4.5}$$

Returning now to 4.5, we find

$$
\begin{aligned}
A = & \sum_{i,j=1}^{n} \int_U a^{ij,h} D_k^h u_{x_i} D_k^h u_{x_j} \zeta^2 dx \\
& + \sum_{i,j=1}^{n} \int_U \left[a^{ij,h} D_k^h u_{x_i} D_k^h u 2\zeta\zeta_{x_j} + \left(D_k^h a^{ij}\right) u_{x_i} D_k^h u_{x_j} \zeta^2 \right. \\
& \left. + \left(D_k^h a^{ij}\right) u_{x_i} D_k^h u 2\zeta\zeta_{x_j} \right] dx \\
= & A_1 + A_2.
\end{aligned}
$$

The uniform ellipticity condition implies

$$
A_1 \geq \theta \int_U \zeta^2 \left| D_k^h Du \right|^2 dx
$$

We want to bound

$$
\underbrace{a^{ij,h} D_k^h u_{x_i} D_k^h u 2\zeta\zeta_{x_j}}_{I_1} + \underbrace{\left(D_k^h a^{ij}\right) u_{x_i} D_k^h u_{x_j} \zeta^2}_{I_2} + \underbrace{\left(D_k^h a^{ij}\right) u_{x_i} D_h^k u 2\zeta\zeta_{x_j}}_{I_3}
$$

We can drop the summation signs and the integrals since they are not playing any role right now. Now, we know that $a^{ij} \in C^1(U)$ with implies uniform bounds on $a^{ij,h}$ and $D_k^h a^{ij}$ (by Section 5.8.2 in Evans' book). Also, $0 \leq \zeta \leq 1$ (which implies $\zeta^2 \leq \zeta$) and $|\nabla \zeta|$ is of the order of $\frac{1}{\operatorname{dist}(V, \mathbb{R}^n \setminus W)}$. And of course, $|D_k^h u_{x_i}| \leq |D_k^h Du|$. This readily yields

$$
|I_1| \leq C_1 \zeta |D_k^h Du| |D_k^h u|, \quad |I_2| \leq C_2 \zeta |D_k^h Du| |Du|, \quad |I_3| \leq C_3 \zeta |D_k^h Du| |Du|
$$

where $C_k, k = 1, 2, 3$ are universal constants. Then you just take $C = \max_{k=1,2,3}\{C_k\}$ sum over i, j and integrate to get the desired bound. Note that

$$
\begin{aligned}
& \zeta |D_k^h Du| |D_k^h u| + \zeta |D_k^h Du| |Du| + \zeta |D_k^h u| |Du| \\
& \leq (\zeta |D_k^h Du| + \zeta |D_k^h u|)(|D_k^h u| + |Du|)
\end{aligned}
$$

Then one can apply Cauchy's theorem or its ϵ-version $ab = (a\sqrt{\epsilon})(b/\sqrt{\epsilon}) \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2$ and $(a + b)^2 \leq C(a^2 + b^2)$ to get

$$
|A_2| \leq \epsilon \int_W \zeta^2 (|D_k^h Du|^2 + |D_k^h u|^2) \, dx + \frac{C}{\epsilon} \int_W (|D_k^h u|^2 + |Du|^2) \, dx.
$$

Now you can absorb the second term in the first integral into the first term of the second integral. Since ζ is supported in W, you only really need to integrate over W. Expanding W to U after all is done is a trivial step.

$$
|A_2| \leq \epsilon \int_U \zeta^2 \left| D_k^h Du \right|^2 dx + \frac{C}{\epsilon} \int_W \left| D_k^h u \right|^2 + |Du|^2 dx
$$

We choose $\epsilon = \frac{\theta}{2}$ and further recall the estimate

$$\int_W \left|D_k^h u\right|^2 dx \le C \int_U |Du|^2 dx$$

thereby obtaining the inequality

$$|A_2| \le \frac{\theta}{2} \int_U \zeta^2 \left|D_k^h Du\right|^2 dx + C \int_U |Du|^2 dx$$

This estimate implies

$$A \ge \frac{\theta}{2} \int_U \zeta^2 \left|D_k^h Du\right|^2 dx - C \int_U |Du|^2 dx$$

Estimate of B. We estimate

$$|B| \le C \int_U (|f| + |Du| + |u|)|v| dx$$

Now

$$\begin{aligned} \int_U |v|^2 dx &\le C \int_U \left|D\left(\zeta^2 D_k^h u\right)\right|^2 dx \\ &\le C \int_W \left|D_k^h u\right|^2 + \zeta^2 \left|D_k^h Du\right|^2 dx \\ &\le C \int_U |Du|^2 + \zeta^2 \left|D_k^h Du\right|^2 dx \end{aligned}$$

$$|B| \le C \int_U (|f| + |Du| + |u|)\, |v|\, dx = C \left(\int_U |f||v| + \int_U |Du||v| + \int_U |u||v| \right). \tag{4.6}$$

Applying Young's inequality $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ to each term in (4.6) and absorbing constants into C, we get

$$|B| \le C\varepsilon \int_U |v|^2\, dx + \frac{C}{\varepsilon} \int_U (f^2 + |Du|^2 + u^2)\, dx. \tag{4.7}$$

Now, recall $v = -D_k^{-h}(\zeta^2 D_k^h u)$. Consider,

$$v(x) = -D_k^{-h} w(x) = -\frac{w(x) - w(x - he_k)}{h} = -\int_0^1 \partial_k w(x - the_k)\, dt \quad \text{for a.e. } x \in U. \tag{4.8}$$

By Jensen's inequality,

$$|D_k^{-h} w(x)|^2 = \left| \int_0^1 \partial_k w(x - the_k)\, dt \right|^2 \le \int_0^1 \left|\partial_k w(x - the_k)\right|^2 dt. \tag{4.9}$$

Integrating (4.9) over U and applying Tonelli,

$$\int_U |D_k^{-h} w(x)|^2\,dx \le \int_U \int_0^1 \left|\partial_k w(x - the_k)\right|^2 dt\,dx = \int_0^1 \int_U \left|\partial_k w(x - the_k)\right|^2 dx\,dt. \tag{4.10}$$

For each fixed $t \in [0,1]$, use the change of variables $y = x - the_k$:

$$\int_U \left|\partial_k w(x - the_k)\right|^2 dx = \int_{U - the_k} \left|\partial_k w(y)\right|^2 dy. \tag{4.11}$$

Since $\operatorname{spt} w \subset W \Subset U$ and $|h| < \frac{1}{2}\operatorname{dist}(W, \partial U)$, we have $\operatorname{spt}(\partial_k w) \subset W \subset U \cap (U - the_k)$; hence

$$\int_{U - the_k} \left|\partial_k w(y)\right|^2 dy = \int_U \left|\partial_k w(y)\right|^2 dy. \tag{4.12}$$

Hence, we get

$$\int_U |D_k^{-h} w|^2\,dx \le \int_0^1 \int_U |\partial_k w(y)|^2\,dy\,dt = \int_U |\partial_k w(y)|^2\,dy \le \int_U |Dw(y)|^2\,dy. \tag{4.13}$$

With $v = -D_k^{-h} w$, (4.13) gives

$$\int_U |v|^2\,dx \;\le\; \int_U |D(\zeta^2 D_k^h u)|^2\,dx, \tag{4.14}$$

Using standard difference-quotient and product-rule estimates,

$$\begin{aligned}\int_U |v|^2\,dx &\le C \int_U \left|D(\zeta^2 D_k^h u)\right|^2 dx \\ &\le C \int_W \left(|D_k^h u|^2 + \zeta^2 |D_k^h Du|^2\right) dx \\ &\le C \int_U \left(|Du|^2 + \zeta^2 |D_k^h Du|^2\right) dx. \end{aligned} \tag{4.15}$$

Substituting (4.15) into (4.7), we find

$$\begin{aligned}|B| &\le C\varepsilon \int_U \left(|Du|^2 + \zeta^2 |D_k^h Du|^2\right) dx + \frac{C}{\varepsilon} \int_U (f^2 + |Du|^2 + u^2)\,dx \\ &\le \varepsilon \int_U \zeta^2 |D_k^h Du|^2\,dx + \frac{C}{\varepsilon} \int_U (f^2 + u^2)\,dx + \frac{C}{\varepsilon} \int_U |Du|^2\,dx. \end{aligned} \tag{4.16}$$

Equation (4.16) is the desired estimate for B. Select $\epsilon = \frac{\theta}{4}$, to obtain

$$|B| \le \frac{\theta}{4} \int_U \zeta^2 \left|D_k^h Du\right|^2 dx + C \int_U f^2 + u^2 + |Du|^2 dx$$

We finally discover

$$\int_V \left|D_k^h Du\right|^2 dx \leq \int_U \zeta^2 \left|D_k^h Du\right|^2 dx \leq C \int_U f^2 + u^2 + |Du|^2 dx$$

for $k = 1, \ldots, n$ and all sufficiently small $|h| \neq 0$.

Theorem 4.2. *Assume* $1 < p < \infty, u \in L^p(V)$, *and there exists a constant* C *such that*

$$\|D^h u\|_{L^p(V)} \leq C$$

for all $0 < |h| < \frac{1}{2} \operatorname{dist}(V, \partial U)$. *Then*

$$u \in W^{1,p}(V), \quad \|Du\|_{L^p(V)} \leq C$$

Using this theorem, we deduce $Du \in H^1_{\text{loc}}(U; \mathbb{R}^n)$, and thus $u \in H^2_{\text{loc}}(U)$, with the estimate

$$\|u\|_{H^2(V)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right)$$

We now refine this estimate by noting that if $V \Subset W \Subset U$, then the same argument shows

$$\|u\|_{H^2(V)} \leq C \left(\|f\|_{L^2(W)} + \|u\|_{H^1(W)}\right) \tag{4.17}$$

for an appropriate constant C depending on V, W, etc. Choose a new cutoff function ζ satisfying

$$\begin{cases} \zeta \equiv 1 \text{ on } W, \ \operatorname{spt} \zeta \subset U, \\ 0 \leq \zeta \leq 1. \end{cases}$$

Now set $v = \zeta^2 u$ in identity 4.3 and perform similar[1] calculations, to discover

$$\int_U \zeta^2 |Du|^2 dx \leq C \int_U f^2 + u^2 dx$$

Thus

$$\|u\|_{H^1(W)} \leq C \left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right).$$

Use this inequality in 4.17 yield 4.2. □

In a similar fashion, we can demystify the higher boundary regularity.

5 Conclusion

Hodge theory sits at a remarkable crossroads between algebraic geometry, differential geometry, and topology. Had we had more time for this project, we

[1] here you will get the full calculation `https://math.stackexchange.com/a/5090822/803654`

would have further explored the algebraic viewpoint, particularly the recent work [NS17]. In summary, Hodge theory exemplifies one of the central themes of modern mathematics: the deep unity between analysis, geometry, and algebra. Its techniques and ideas will continue to influence future developments across many fields, making it an essential topic for anyone seeking a deeper understanding of geometric structures.

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Approval

The internship report titled "Understanding the Hodge Decomposition and Elliptic Regularity Theorem" submitted by Emon Hossain, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfillment of the requirements of the internship program. The internship was conducted under the supervision of **Siddiqur Rahman Milon** during the period **15 July 2025** to **15 October 2025**.

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