Approval

The internship report titled “**Derivation of equations of motion and solutions in time dependent Kasner brane backgrounds**” submitted by **Disha Sen**, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfilment of the requirements of the internship program.

The internship was conducted under the supervision of **Ratul Mahanta and Ahmed Rakin Kamal** during the period **15 July 2025** to **15 October 2025**.

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Internship Report

on

Derivation of eq. of motion and solutions in time-dependent kasner brane backgrounds

(Based on the work of W. A. Sabra)

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Introduction:

The internship report presents a study of Kasner type solution in higher dimensional gravity theories, including the effects of scalar (dilation) and m form fields. The primary objective was to derive the equation of motions from the actions presented in equations (1.10), (1.18) and (2.26) of Sabra's paper and then the consistency conditions were analyzed and the corresponding solutions were achieved from the metric ansatz for the action (2.26) including a dilation scalar field ϕ and an antisymmetric m form field strength F_m. Generally time dependent backgrounds in gravitational theories are fundamental for understanding the dynamics of early universe, anisotropic expansion and the nature of cosmological singularities. The Kasner metric plays an important role in theoretical cosmology

In higher dimensional gravity theories, Kasner type solutions naturally extend to models with scalar fields (dilation) and anti-symmetric field strengths (n forms). These solutions often called toy models help to investigate cosmological singularities relevent to supergravity and string theory. The internship tasks involved deriving these time dependent solutions, after deriving ricci tensor componnents and equations of motion. The

The study aimed to provide practical understanding of variational principles and a theoritical appreciation of time dependent kasner type solutions in higher dimensional gravity. To reach our ultimate goal we went through solving useful identities. (a) $d^2 F_p = 0$. (b) $(**\omega)_{i_1 \ldots i_p} = (-1)^{p(D-p)+s} \omega_{i_1 \ldots i_p}$. (c) $(*d*F)_{k_1 \ldots k_{p-1}} = (-1)^{s+p(d-p)} \nabla_\sigma F^{\sigma}{}_{k_1 \ldots k_{p-1}}$ (d) $\langle d\eta, \omega \rangle = \langle \eta, d^{+}\omega \rangle$. (e) Bianchi identity. $dF_p = 0$. (f) writing E.H action in form notation that are also discussed in the main text. Each action described in the main text reveals a different layer of physical inter action from the geometric dynamic of spacetime to the influence of matter and fluxes on cosmic evolution.

Main text:

* Derive Riemann curvature tensor and Ricci tensor from the Kasner metric.

⇒ The Kasner metric

$$ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2.$$

The metric tensor $g_{\mu\nu}$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & t^{2p_1} & 0 & 0 \\ 0 & 0 & t^{2p_2} & 0 \\ 0 & 0 & 0 & t^{2p_3} \end{bmatrix}$$

$$g^{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & t^{-2p_1} & 0 & 0 \\ 0 & 0 & t^{-2p_2} & 0 \\ 0 & 0 & 0 & t^{-2p_3} \end{bmatrix}$$

christoffel symbol formula:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right)$$

$g_{00} = 1$. $g_{11} = t^{2P_1}$

$g_{22} = t^{2P_2}$, $g_{33} = t^{2P_3}$

$$\Gamma^{0}_{00} = \frac{1}{2} g^{00} \left(-\partial_0 g_{00} \right)$$

$$= 0$$

$$\Gamma^{0}_{11} = \frac{1}{2} g^{00} \left(-\partial_0 g_{11} \right)$$

$$= \frac{1}{2} \times -2P_1 \times t^{2P_1 - 1} \times (-1)$$

$$= P_1 t^{2P_1 - 1}$$

$$\Gamma^{0}_{22} = P_2 t^{2P_2 - 1}$$

$$\Gamma^{0}_{33} = P_3 \, t^{2P_3 - 1}$$

$$\Gamma^{1}_{10} = \frac{1}{2} \times g^{11} \times \partial_0 g_{11}$$

$$= \frac{1}{2} \times t^{-2P_1} \times 2P_1 \times t^{2P_1 - 1}$$

$$= P_1 \, t^{-1} = \Gamma^{1}_{01}$$

$$\Gamma^{2}_{02} = \Gamma^{2}_{20} = \frac{1}{2} \times t^{-2P_2} \times 2P_2 \times t^{2P_2 - 1}$$

$$= P_2 \, t^{-1}$$

$$\Gamma^{3}_{03} = \Gamma^{3}_{30} = \frac{1}{2} \times t^{-2P_3} \times 2P_3 \times t^{2P_3 - 1}$$

$$= P_3 \, t^{-1}$$

Riemann curvature tensors:

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma}$$

$$- \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma}$$

$$R^{1}_{010} = -\partial_0 \Gamma^{1}_{10} - \Gamma^{1}_{01}\Gamma^{1}_{10}$$

$$= P_1 t^{-2} - (P_1 t^{-1})(P_1 t^{-1})$$

$$= P_1(1 - P_1) t^{-2}$$

$$R^{2}_{020} = -\partial_0 \Gamma^{2}_{20} - \Gamma^{2}_{02}\Gamma^{2}_{20}$$

$$= P_2 t^{-2} - (P_2 t^{-1})(P_2 t^{-1})$$

$$= P_2(1 - P_2) t^{-2}.$$

$$R^{3}_{030} = -\partial_0 \Gamma^{3}_{30} - \Gamma^{3}_{03}\Gamma^{3}_{30}$$

$$= P_3 t^{-2} - (P_3 t^{-1})(P_3 t^{-1})$$

$$= P_3(1 - P_3) t^{-2}$$

$$\therefore R^{0}_{i0i} = P_i(P_i - 1) t^{2P_i - 2}$$

where, $i = 1, 2, 3$

$$R^{1}_{212} = (p_1 t^{-1})(p_2 t^{2p_2-1})$$

$$= p_1 p_2 t^{2p_2-2} = R^{2}_{121}$$

$$R^{1}_{313} = \Gamma^{1}_{10}\Gamma^{0}_{33} = (p_1 t^{-1})(p_3 t^{2p_3-1})$$

$$= p_1 p_3 t^{2p_3-2}$$

$$R^{2}_{121} = \Gamma^{2}_{20}\Gamma^{0}_{11} = (p_2 t^{-1})(p_1 t^{2p_1-1})$$

$$= p_1 p_2 t^{2p_1-2}$$

$$R^{2}_{323} = \Gamma^{2}_{20}\Gamma^{0}_{33} = (p_2 t^{-1})(p_3 t^{2p_3-1})$$

$$= p_2 p_3 t^{2p_3-2}$$

$$R^{3}_{131} = \Gamma^{3}_{30}\Gamma^{0}_{11} = (p_3 t^{-1})(p_1 t^{2p_1-1})$$

$$= p_1 p_3 t^{2p_1-2}$$

$$R^{3}_{232} = \Gamma^{3}_{30}\Gamma^{0}_{22} = (p_3 t^{-1})(p_2 t^{2p_2-1})$$

$$= p_2 p_3 t^{2p_2-}$$

$$R^1_{020} = R^1_{030} = R^2_{010} = R^1_{123} = R^1_{132}$$

$$= R^2_{013} = R^3_{012} = R^0_{123} = R^0_{132}$$

$$= R^0_{011} = R^0_{022} = R^0_{033} = R^1_{111} = R^3_{333} = 0$$

Ricci tensor:

Ricci tensor is obtained by contracting the first and third indices of the Riemann tensor.

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$$

Here, λ runs over all coordinate indices. 0, 1, 2, 3.

R for $u=0, v=1$.

$$R_{01} = R^{0}_{001} + R^{1}_{011} + R^{2}_{021} + R^{3}_{031}$$

$$= 0$$

$\therefore R_{uv} = 0$ when $u \neq v$.

$$R_{00} = R^{0}_{000} + R^{1}_{010} + R^{2}_{020} + R^{3}_{030}$$

$$= 0 + P_1(1-P_1)\ell^{-2} + P_2(1-P_2)\ell^{-2} + P_3(1-P_3)\ell^{-2}$$

$$= \frac{P_1(1-P_1) + P_2(1-P_2) + P_3(1-P_3)}{\ell^2}$$

$$R_{11} = R^{0}_{101} + R^{1}_{111} + R^{2}_{121} + R^{3}_{131}$$

$$= p_1 (p_1 - 1) t^{2p_1 - 2} + p_1 p_2 t^{2p_1 - 2} + p_1 p_3 t^{2p_1 - 2}$$

$$R_{22} = R^{0}_{202} + R^{2}_{222} + R^{3}_{232} + R^{1}_{212}$$

$$= p_2 (p_2 - 1) t^{2p_2 - 2} + p_2 p_3 t^{2p_2 - 2} + p_1 p_2 t^{2p_2 - 2}$$

$$R_{33} = R^{0}_{303} + R^{3}_{333} + R^{1}_{313} + R^{2}_{323}$$

$= P_3(P_3-1)\, t^{2P_3-2} + P_1P_3\, t^{2P_3-2}$

$+ P_2P_3\, t^{2P_3-2}$.

21 Prove $d^2F_p = 0$. (i.e. if we apply the exterior derivatives twice on a p form we get zero [certain smoothness on the components of F_p are assumed]

$\Rightarrow F = \frac{1}{p!} F_{\mu_1 \ldots \mu_p}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$

$(dF)_{\mu_0 \ldots \mu_p}$

$= \partial_{\mu_0} F_{\mu_1 \ldots \mu_p} - \partial_{\mu_1} F_{\mu_0 \mu_2 \ldots \mu_p}$

$+ \partial_{\mu_2} F_{\mu_0 \mu_1 \ldots \mu_p} - \ldots + (-1)^p \partial_{\mu_p} F_{\mu_0 \ldots \mu_{p-1}}$

Now

$$(d^2F)_{\mu_0 \ldots\ldots \mu_{p+1}}$$

$$= \partial_{\mu_0} (dF)_{\mu_1 \ldots\ldots \mu_{p+1}}$$

$$- \partial_{\mu_1} (dF)_{\mu_0 \mu_2 \ldots\ldots \mu_{p+1}} + \cdots$$

$$+ (-1)^{p+1} \partial_{\mu_{p+1}} (dF)_{\mu_0 \ldots\ldots \mu_p}$$

If we consider the first term

$$\partial_{\mu_0} (dF)_{\mu_1 \ldots\ldots \mu_{p+1}}$$

$$= \partial_{\mu_0} \Big[\partial_{\mu_1} F_{\mu_2 \ldots\ldots \mu_{p+1}} - \partial_{\mu_2} F_{\mu_1 \mu_3 \ldots \mu_{p+1}}$$

$$+ \ldots\ldots + (-1)^p \partial_{\mu_{p+1}} F_{\mu_1 \ldots\ldots \mu_p} \Big]$$

Similarly for the second term

Now each term of d^2F is of the form: $\partial_{\mu_i} \partial_{\mu_j} F$

Partial derivatives commute with each other: $\partial_{\mu_i} \partial_{\mu_j} = \partial_{\mu_j} \partial_{\mu_i}$

Each pair of derivatives due to antisymmetrization gives.

$$\partial_{\mu_i} \partial_{\mu_j} F \ldots - \partial_{\mu_j} \partial_{\mu_i} F \ldots = 0$$

As a result every term cancels in pairs.

$$\therefore \cancel{d^2F} \quad (d^2F)_{\mu_0 \ldots \mu_{p+1}} = 0$$

31 If you take Hodge dual twice on a p form you get the same p form back upto a sign. Find the sign (which ~~does~~ depends on the signature of the metric the rank of the form p and the dimension of the manifold.)

$$\Rightarrow \omega = \frac{1}{p!} \omega_{i_1 \dots i_p} dx^{i_1} \wedge \dots dx^{i_p}$$

$$(*\omega)_{j_1 \dots j_{D-p}}$$

$$= \frac{\sqrt{|g|}}{p!(D-p)!} \epsilon_{j_1 \dots j_{D-p} i_1 \dots i_p} \omega^{i_1 \dots i_p}$$

$(**\omega)_{i_1 \dots i_p}$

$$= \frac{\sqrt{|g|}}{p!} \epsilon_{i_1 \dots i_p \dot{a}_1 \dots \dot{a}_{D-p}} (*\omega)^{\dot{a}_1 \dots \dot{a}_{D-p}}$$

$$= \frac{\sqrt{|g|}}{p!} \epsilon_{i_1 \dots i_p \dot{a}_1 \dots \dot{a}_{D-p}} \times \frac{\sqrt{|g|}}{p!(D-p)!} \epsilon^{\dot{a}_1 \dots \dot{a}_{D-p} k_1 \dots k_p} \omega_{k_1 \dots k_p}$$

$$= \frac{|g|}{(p!)^2 (D-p)!} \epsilon_{i_1 \dots i_p \dot{a}_1 \dots \dot{a}_{D-p}} \epsilon^{\dot{a}_1 \dots \dot{a}_{D-p} k_1 \dots k_p} \omega_{k_1 \dots k_p}$$

$$= \cancel{(-1)^p} ($$

$$= \frac{1}{(p!)^2 (D-p)!} \times (-1)^s \times (-1)^{p(D-p)} \, p!(D-p)! \, \delta^{k_1 \dots k_p}_{i_1 \dots i_p} \omega_{k_1 \dots k_p}$$

$$= \frac{1}{(p!)^2 (D-p)!} \times (-1)^{p(D-p)+s} (p!)^2 (D-p)! \, \omega_{i_1 \dots i_p}$$

$= (-1)^{P(D-P)+S}\ \omega_{i_1 \dots i_P}$

41 convince yourself that taking Hodge dual then exclesion derivative then again a Hodge dual on F_P gives divergence where D stands for covarient derivative the repeated indices "m" are contracted - other indices n are free upto a sign.

$$F^{(p)} = \frac{1}{p!} F_{\nu_1 \dots \nu_p} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_p}$$

$$(*F)_{\mu_1 \dots \mu_{d-p}} = \frac{1}{p!} \epsilon_{\mu_1 \dots \mu_{d-p} \nu_1 \dots \nu_p} F^{\nu_1 \dots \nu_p}$$

where, $F^{\nu_1 \dots \nu_p} = \theta^{\nu_1 a_1} \dots \theta^{\nu_p a_p} F_{a_1 \dots a_p}$

$(d*F)_{\mu_0 \dots \mu_{d-p}}$

$$= \partial_{\mu_0} (*F)_{\mu_1 \dots \mu_{d-p}} - \partial_{\mu_1} (*F)_{\mu_0 \dots \mu_{d-p}}$$

$$+ \dots \dots + (-1)^{D-p+1} \partial_{D-p+1} (*F)_{\mu_0 \dots \mu_{d-p-1}}$$

If we apply hodge dual star $(*)$ on $(d*F)_{\mu_0 \dots \mu_{d-p}}$, ~~[illegible]~~

$(*d*F)_{k_1 \ldots k_{p-1}}$

$$= \frac{1}{(p-1)!} \epsilon_{k_1 \ldots k_{p-1} \mu_0 \ldots \mu_{D-p}} (d*F)^{\mu_0 \ldots \mu_{D-p}}$$

$= \frac{1}{(p-}$ If we consider the first term

first term of $(*d*F)_{k_1 \ldots k_{p-1}}$

$$= \frac{1}{(p-1)!} \epsilon_{k_1 \ldots k_{p-1} \mu_0 \ldots \mu_{d-p}} \times$$

$$\partial_{\mu_0} (*F)^{\mu_1 \ldots \mu_{d-p}}$$

$$= \frac{1}{(p-1)!} \epsilon_{k_1 \ldots k_{p-1} \underline{\mu_0 \ldots \mu_{d-p}}} \times$$

$$\partial_{\mu_0} \times \frac{1}{p!} \epsilon^{\mu_1 \ldots \mu_{d-p} v_1 \ldots v_p} F_{v_1 \ldots v_p}$$

$$= \frac{1}{(p-1)!\,p!} \times \delta^{v_1 \ldots v_p}_{k_1 \ldots k_{p-1} \mu_0} (-1)^{s} (d-p)! (-1)^{p(d-p)}$$

$$\times p! \times \partial_{\mu_0} F_{v_1 \ldots v_p}$$

$$= \frac{1}{(p-1)!\,p!} \times (p-1)! \times p!\, (-1)^{s+p(d-p)}$$

$$\times\, \partial_{\mu_0} F_{\kappa_1 \ldots \kappa_{p-1} \mu_0}$$

$$[\ \cancel{5}.\ \delta^{\nu_1 \ldots \nu_p}_{\kappa_1 \ldots \kappa_{p-1} \mu_0} F_{\nu_1 \ldots \nu_p}$$

$$= F_{\kappa_1 \ldots \kappa_{p-1} \mu_0}]$$

$$\cancel{= F_{\kappa_1 \ldots \kappa_p}]}$$

we can write

$$(*d*F)_{\kappa_1 \ldots \kappa_{p-1}}$$

$$= (-1)^{s+p(d-p)} \cdot \nabla_\sigma F^{\sigma}{}_{\sigma \kappa_1 \ldots \kappa_{p-1}}$$

where $\sigma = \mu_0, \ldots, \cancel{\mu_{d-p+1}}\ \mu_{d-p+1}$

$*\ \langle d\eta, \omega \rangle = \langle \eta, d^{+}\omega \rangle$

$\Rightarrow$

We know.

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta.$$

$\therefore \langle d\eta, \omega \rangle = \int_M d\eta \wedge *\omega$

$d(\eta \wedge *\omega) = d\eta \wedge *\omega + (-1)^{p-1} \eta \wedge d(*\omega)$

or, $d\eta \wedge *\omega = d(\eta \wedge *\omega) - (-1)^{p-1} \eta \wedge d(*\omega)$

or, $\int_M d\eta \wedge *\omega = \int_M d(\eta \wedge *\omega)$

$\quad - (-1)^{p-1} \int_M \eta \wedge d(*\omega)$

By stokes theorem.

$$\int_M d(\eta \wedge *\omega) = \int_{\partial M} \eta \wedge *\omega$$

So,

$\langle d\eta, \omega \rangle = \int_{\partial M} \eta \wedge * \omega + (-1)^p \int_M \eta \wedge d * \omega$

vanishing the boundary term,

$$\int_{\partial M} \eta \wedge * \omega = 0$$

$\therefore \langle d\eta, \omega \rangle = (-1)^p \int_M \eta \wedge d * \omega$

again, $\langle \eta, d^{+}\omega \rangle = \int_M \eta \wedge * (d^{+}\omega)$

$= \int_M \eta \wedge * \left[(-1)^{Dp+D+1} * d * \omega \right]$

$= (-1)^{Dp+D+1} \int_M \eta \wedge * * d * \omega$

$= (-1)^p \int_M \eta \wedge d * \omega.$

$\therefore \langle \eta, d^{+}\omega \rangle = \langle d\eta, \omega \rangle$

(b) Write Einstein hilbert action in form notation

$\Rightarrow$ $$S_{EH} = \frac{1}{8\pi G} \int d^4x \sqrt{-g}\, R$$

$$= \frac{1}{8\pi G} \int R * 1$$

(c) Action of a p form field strength and it's equation of motion and Bianchi identity

$\Rightarrow$ Let A_{p-1} be a $p-1$ form potential.

$$\therefore F_p = dA_{p-1}$$

$$\therefore S = -\frac{1}{2} \int F_p \wedge * F_p$$

$$= -\frac{1}{2p!} \int d^D x \sqrt{-g}\, F_{\mu_1 \ldots \mu_p} F^{\mu_1 \ldots \mu_p}$$

Since, $F_p = dA_{p-1}$

$$\delta F_p = d(\delta A_{p-1})$$

$$\delta S = -\frac{1}{2}\int (d(\delta A_{p-1}) \wedge * F_p +$$

$$F_p \wedge * d(\delta A_{p-1})$$

Here the both terms are equal.

$$\delta S = -\int d(\delta A_{p-1}) \wedge * F_p$$

Using stoke's theorm and assuming boundary term vanish.

$$\delta S = -(-1)^p \int \delta A_{p-1} \wedge d(*F_p)$$

The action is stationary $(\delta S = 0)$.

$$\therefore d(*F_p) = 0$$

Now, $F_p = dA_{p-1}$

The Bianchi identity is

$$dF_p = d(dA_{p-1}) = 0, \therefore dF_p = 0$$

Deriving equation of motion from actions

(1.10) $S = \int d^dx \sqrt{|g|} \left(R - \frac{\epsilon}{2m!} F_{\mu_1 \ldots \mu_m} F^{\mu_1 \ldots \mu_m}\right)$

$= \int \left(R*1 - \frac{\epsilon}{2} F \wedge *F\right).$

$\delta(\sqrt{|g|} R) = (\delta\sqrt{|g|})R + \sqrt{|g|}\,\delta R$

$\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}\, g^{\mu\nu} \delta g_{\mu\nu}$

$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}$

$[R = g^{\mu\nu} R_{\mu\nu}]$

$\therefore \delta(\sqrt{|g|} R)$

$= \frac{1}{2}\sqrt{|g|}\, g^{\mu\nu} R\, \delta g_{\mu\nu}$

$+ \sqrt{|g|}\, R_{\mu\nu}\, \delta g^{\mu\nu}$ + boundary term

$= \sqrt{|g|}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) \delta g^{\mu\nu}$

+ boundary term.

$$= \sqrt{|g|}\, G_{\mu\nu}\, \delta g^{\mu\nu} + \partial_a(\ldots)$$

$$\delta\left(-\frac{\varepsilon}{2m!}\sqrt{|g|}\, F_{\mu_1 \ldots \mu_m} F^{\mu_1 \ldots \mu_m}\right)$$

$$= -\frac{\varepsilon}{2m!}\, \delta\left(\sqrt{|g|}\, F^2\right)$$

$$\delta\left(\sqrt{|g|}\, F^2\right) = \left(\delta\sqrt{|g|}\right) F^2 + \sqrt{|g|}\, \delta(F^2)$$

$$= -\frac{1}{2}\sqrt{|g|}\, g_{\mu\nu} F^2 \delta g^{\mu\nu} + \sqrt{|g|}\, \delta(F^2)$$

$$\delta(F^2) = \delta\left(F_{\mu_1 \ldots \mu_m} F^{\mu_1 \ldots \mu_m}\right)$$

$$= F_{\mu_1 \ldots \mu_m}\, \delta F^{\mu_1 \ldots \mu_m}$$

Now,

$$F^{\mu_1 \ldots \mu_m} = g^{\mu_1 a_1} g^{\mu_2 a_2} \ldots g^{\mu_m a_m} F_{a_1 \ldots a_m}$$

$$\delta(F^2) = F_{\mu_1 \ldots \mu_m} \sum_{k=1}^{m} \left(g^{\mu_1 a_1} \ldots \delta g^{\mu_k a_k} \ldots g^{\mu_m a_m}\right) F_{a_1 \ldots a_m}$$

if I consider one term ($K=1$)

$\mu_1 \to \mu$, $a_1 \to \nu$

then the first term becomes

$$F_{\mu a_2 \ldots a_m} F_{\nu}{}^{a_2 \ldots a_m} \delta g^{\mu\nu}$$

$$\therefore \delta(F^2) = m F_{\mu a_2 \ldots a_m} F_{\nu}{}^{a_2 \ldots a_m} \delta g^{\mu\nu}$$

δS

$$= \int d^d x \ \delta(\sqrt{|g|} R)$$

$$- \int d^d x \frac{\epsilon}{2m!} \ \delta(\sqrt{|g|} F^2)$$

$$= \int d^d x \ G_{\mu\nu} \delta g^{\mu\nu} \sqrt{|g|}$$

$$+ \int d^d x \frac{\epsilon}{2m!} \frac{1}{2} \sqrt{|g|} \ g_{\mu\nu} F^2 \delta g^{\mu\nu}$$

$$- \int d^d x \frac{\epsilon m}{2m!} \sqrt{|g|} \ F_{\mu a_2 \ldots a_m} F_{\nu}{}^{a_2 \ldots a_m} \delta g^{\mu\nu}$$

$$= \int d^d x \, \delta g^{\mu\nu} \sqrt{|g|}$$

$$\left(G_{\mu\nu} + \frac{\epsilon}{4m!} g_{\mu\nu} F^2 - \frac{\epsilon m}{2m!} F_{\mu a_2 \ldots a_m} F_\nu^{\;a_2 \ldots a_m} \right)$$

[boundary terms ~~geo~~ goes to zero]

$\delta S = 0$ according to principle of least action.

$$G_{\mu\nu} = \frac{\epsilon}{2m!} \left(m F_{\mu a_2 \ldots a_m} F_\nu^{\;a_2 \ldots a_m} - \frac{1}{2} g_{\mu\nu} F^2 \right)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = G_{\mu\nu} = T_{\mu\nu}$$

or. $$g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} R = g^{\mu\nu} T_{\mu\nu}$$

or. $$R - \frac{d}{2} R = T$$

$$\therefore \quad R = \frac{2}{2-d} T = -\frac{2}{d-2} T$$

we know,

$$T_{\mu\nu} = -\frac{1}{\sqrt{|g|}} \frac{\delta S_F}{\delta g^{\mu\nu}}$$

or, $$\delta S_F = -\int \sqrt{|g|}\, T_{\mu\nu}\, \delta g^{\mu\nu}$$

$$\therefore \quad -T_{\mu\nu} = \frac{1}{4m!} \epsilon g_{\mu\nu} F^2 - \frac{m\epsilon}{2m!} F_{\mu a_2 \ldots a_m} F_{\nu}^{a_2 \ldots a_m}$$

or, $$T_{\mu\nu} = \frac{m\epsilon}{2m!} F_{\mu a_2 \ldots a_m} F_{\nu}^{\;a_2 \ldots a_m}$$

$$- \frac{1 \cdot \epsilon}{4 m!} g_{\mu\nu} F^2$$

$$= \frac{i\epsilon}{2(m-1)!} F_{\mu a_2 \ldots a_m} F_{\nu}{}^{a_2 \ldots a_m}$$

$$- \frac{\epsilon}{4m!} g_{\mu\nu} F^2$$

$$R = -\frac{2}{d-2} g^{\mu\nu} T_{\mu\nu}$$

$$= -\frac{2}{d-2} g^{\mu\nu} \left[\frac{i\epsilon}{2(m-1)!} F_{\nu a_2 \ldots a_m} F_{\nu}{}^{a_2 \ldots a_m} - \frac{\epsilon}{4m!} g_{\mu\nu} F^2 \right]$$

$$= \frac{2}{2-d} \left[\frac{\epsilon F^2}{2(m-1)!} - \frac{\epsilon d F^2}{4m!} \right]$$

$$= \frac{2}{2-d} \left[\frac{\epsilon m}{2m!} - \frac{\epsilon d}{4m!} \right] F^2$$

$$= \frac{2}{2-d} \times \frac{2m-d}{4m!} \epsilon F^2$$

$$= \frac{d-2m}{2(d-2)m!} \epsilon F^2$$

$$\therefore R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \frac{d-2m}{2(d-2)m!} \epsilon F^2$$

$$= \frac{1}{2(m-1)!} \epsilon F_{\mu a_2 \ldots a_m} F_\nu{}^{a_2 \ldots a_m} - \frac{1}{4m!} g_{\mu\nu} F^2$$

or, $R_{\mu\nu}$

$$= \frac{1}{2(m-1)!} \epsilon F_{\nu a_2 \ldots a_m} F_\nu{}^{a_2 \ldots a_m}$$

$$- \frac{\epsilon}{4m!} g_{\mu\nu} F^2 + \frac{\epsilon}{2} g_{\mu\nu} \frac{d-2m}{2(d-2)m!} F^2$$

$$= \frac{\epsilon}{2(m-1)!} F_{\nu a_2 \ldots a_m} F_\nu{}^{a_2 \ldots a_m}$$

$$- \left[\frac{\epsilon}{4m!} - \frac{\epsilon}{4} \frac{d-2m}{(d-2)m!} \right] F^2 g_{\mu\nu}$$

$$= \frac{\epsilon}{2(m-1)!} F_{\nu a_2 \ldots a_m} F_\nu{}^{a_2 \ldots a_m}$$

$$- \left[\frac{d-2-d+2m}{4(d-2)m!} \right] \epsilon F^2 g_{\mu\nu}$$

$$= \frac{\epsilon}{2(m-1)!} F_{\nu a_2 \ldots a_m} F_{\nu}^{\ a_2 \ldots a_m}$$

$$- \frac{\epsilon(m-1)}{2(d-2)\, m!} F^2 g_{\mu\nu}$$

$$\therefore R_{\mu\nu} - \epsilon\left(\frac{1}{2(m-1)!} F_{\nu a_2 \ldots a_m} F_{\nu}^{\ a_2 \ldots a_m} - g_{\mu\nu} \frac{(m-1)}{2m!\,(d-2)} F^2 \right) = 0.$$

Variation with potential:

$$S_F = -\frac{\epsilon}{2}\int F\wedge *F$$

or, $$\delta S_F = -\frac{\epsilon}{2}\int \delta(F\wedge *F)$$

or, $$\delta S_F = -\frac{\epsilon}{2}\int \delta F\wedge *F$$

Now, $F = dA$

or, $\delta F = d(\delta A)$

$$\therefore \delta S_F = -\frac{\epsilon}{2}\int d(\delta A)\wedge *F$$

$$= -\frac{\epsilon}{2}\int \delta A\wedge d*F$$

Now, $F = \frac{1}{m!} F_{\mu_1 \dots \mu_m} dx^{\mu_1}\wedge \dots \wedge dx^{\mu_m}$

$$(*F)_{\nu_1 \dots \nu_{d-m}} = \frac{1}{m!(d-m)!}\sqrt{|g|}\,\epsilon_{\nu_1 \dots \nu_{d-m}\mu_1 \dots \mu_m} F^{\mu_1 \dots \mu_m}$$

Then, d we know, $d*F = 0$

$$\therefore \partial_\mu\left(\sqrt{|g|}\, F^{\mu\nu_2 \dots \nu_m}\right) = 0$$

(1.18) $S = \int d^d x \sqrt{|g|} \left(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi\right)$.

In form notation.

$$S = \int R * 1 - \frac{1}{2} d\phi \wedge * d\phi.$$

* variation with respect to ϕ:

The ϕ part of Lagrangian is

$$L_\phi = -\frac{1}{2} \sqrt{|g|}\, g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

$$\delta L_\phi = -\frac{1}{2} \sqrt{|g|}\, g^{\mu\nu} \left(\partial_\mu (\delta\phi) \partial_\nu \phi + \partial_\mu \phi\, \partial_\nu (\delta\phi)\right)$$

$$= -\sqrt{|g|}\, g^{\mu\nu} \partial_\mu \phi\, \partial_\nu (\delta\phi)$$

$$\delta S_\phi = -\int d^d x \sqrt{|g|}\, g^{\mu\nu} \partial_\mu \phi\, \nabla_\nu (\delta\phi)$$

$$= -\int d^d x \sqrt{|g|}\, g^{\mu\nu} \nabla_\nu$$

$= \int d^d x \sqrt{|g|} \nabla_\nu (g^{\mu\nu} \partial_\mu \phi) \delta\phi$

[Ingrating by parts and dropping the boun

$\sqrt{|g|}\, \nabla_\nu g^{\mu\nu} \partial_\mu \phi = \nabla^\mu \partial_\mu \phi$

If $\delta S_\phi = 0$

$\nabla^\mu \partial_\mu \phi = 0$ - that gives a scalar motion

* Variation with respectie to the metric:

$S = S_{grav} + S\phi$

$S_{grav} = \int \sqrt{|g|}\, R$

~~$\delta S_{grav} = \delta$~~ Now

$\delta(\sqrt{|g|}\, R) = \sqrt{|g|}\, \delta R + R\, \delta\sqrt{|g|}$

$= \frac{1}{2}\sqrt{|g|}\, g^{\mu\nu} R\, \delta g_{\mu\nu}$

$+ \sqrt{|g|}\, R_{\mu\nu}\, \delta g^{\mu\nu}$ + boundary term

$= \sqrt{|g|}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) \delta g^{\mu\nu}$

+ boundary term

$\delta S_{grav} = \int d^d x \sqrt{|g|}\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right)\delta g^{\mu\nu}$

+ (boundary terms)

$S_\phi = -\frac{1}{2}\int d^d x \sqrt{|g|}\, g^{\mu\nu}\, \partial_\mu \phi\, \partial_\nu \phi$

$\delta S_\phi = -\frac{1}{2}\Big[\int d^d x \Big[(\delta\sqrt{|g|})\, g^{\mu\nu}\, \partial_\mu \phi\, \partial_\nu \phi$

$$+ \sqrt{|g|} \, (\delta g^{\mu\nu}) \, \partial_\mu \phi \, \partial_\nu \phi]$$

$$= -\frac{1}{2} \int d^d x \sqrt{|g|} \left[-\frac{1}{2} g_{\mu\nu} (\partial\phi)^2 \right.$$

$$\left. + \partial_\mu \phi \, \partial_\nu \phi \right] \delta g^{\mu\nu}$$

$$\cancel{= -\frac{1}{2} \int d^d x \sqrt{|g|} \left(\partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial\phi)^2 \right)}$$

$$= -\frac{1}{2} \int d^d x \sqrt{|g|} \left(\partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial\phi)^2 \right) \delta g^{\mu\nu}$$

$$T_{\mu\nu} = -\frac{1}{\sqrt{|g|}} \frac{\delta S_\phi}{\delta g^{\mu\nu}}$$

$$= \frac{1}{2} \left(\partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial\phi)^2 \right)$$

b $\delta S_\phi = 0$

$$\therefore G_{\mu\nu} = T_{\mu\nu} = \frac{1}{2} \left(\partial_\mu \phi \, \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial\phi)^2 \right)$$

(2.26) $S = \int d^d x \sqrt{|g|} \left(R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\epsilon}{2m!} e^{\beta \phi} F_m^2 \right)$

$= \int \left[R * 1 - \frac{1}{2} d\phi \wedge * d\phi - \frac{1}{2} e^{\beta\phi} F_m \wedge * F_m \right]$

Varying with metric,

$\delta S = \int d^d x \sqrt{|g|} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu}$

$- \left[\frac{1}{2} \sqrt{|g|} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} g_{\mu\nu} (\partial\phi)^2 \right] \delta g^{\mu\nu} d^d x$

$+ \left[d^d x \left(\frac{\epsilon}{2m!} \frac{1}{2} \sqrt{|g|} g_{\mu\nu} F^2 - \frac{\epsilon m}{2m!} \sqrt{|g|} F_{\mu a_2 \dots a_m} F_\nu{}^{a_2 \dots a_m} \right) e^{\beta\phi} \delta g^{\mu\nu} \right]$

$R^{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} g_{\mu\nu} (\partial\phi)^2 - \frac{\epsilon e^{\beta\phi}}{2m!} \frac{1}{2} g_{\mu\nu} F^2 + \frac{\epsilon e^{\beta\phi}}{2(m-1)!} F_{\mu a_2 \dots a_m} F_\nu{}^{a_2 \dots a_m}$

$$\therefore T_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}(\partial\phi)^2$$

$$- \frac{\epsilon e^{\beta\phi}}{4m!} g_{\mu\nu} F^2 + \frac{\epsilon e^{\beta\phi}}{2(m-1)!} F_{\mu a_2 \ldots a_m} F_\nu{}^{a_2 \ldots a_m}$$

$$\therefore R = -\frac{2}{d-2} g^{\mu\nu} T_{\mu\nu}$$

$$= -\frac{2}{d-2} g^{\mu\nu} \left[\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}(\partial\phi)^2 \right.$$

$$\left. - \frac{\epsilon e^{\beta\phi}}{4m!} g_{\mu\nu} F^2 + \frac{\epsilon e^{\beta\phi}}{2(m-1)!} F_{\mu a_2 \ldots a_m} F_\nu{}^{a_2 \ldots a_m} \right]$$

$$= -\frac{1}{d-2} g^{\mu\nu} \partial_\mu\phi\partial_\nu\phi$$

$$+ \frac{1}{d-2}\frac{1}{2} d (\partial\phi)^2 + \frac{\epsilon e^{\beta\phi}}{2m!(d-2)} d F^2$$

$$- \frac{\epsilon e^{\beta\phi}}{(d-2)(m-1)!} g^{\mu\nu} F_{\mu a_2 \ldots a_m} F_\nu{}^{a_2 \ldots a_m}$$

$$= \frac{1}{d-2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi$$

$$+ \frac{1}{2(d-2)} d (\partial\phi)^2 + \frac{\epsilon e^{\beta\phi}}{2m!(d-2)} d F^2$$

$$- \frac{\epsilon e^{\beta\phi}}{(d-2)(m-1)!} F^2$$

$$= \frac{1}{d-2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi$$

$$+ \frac{1}{2(d-2)} d (\partial\phi)^2 + \left(\frac{\epsilon e^{\beta\phi} d}{2m!(d-2)} - \frac{\epsilon e^{\beta\phi} m}{(d-2)(m-1)!} \right) F^2$$

$$= \cancel{\left(\frac{1}{d-2}\right)} = \frac{1}{d-2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi$$

$$\cancel{+ \frac{1}{2(d-2)}} d \cancel{(\partial\phi)^2 + \left(\frac{}{2(d-2) m!(m-1)!} \right.}$$

$\frac{1}{2} g_{\mu\nu}$

$$= \frac{1}{d-2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi + \frac{1}{2(d-2)} d (\partial\phi)^2$$

$$+ \left(\frac{\epsilon e^{\beta\phi} d - 2\epsilon e^{\beta\phi} \cancel{(m-1)!}\, m}{2m!(d-2)} \right) F^2 -$$

$$\therefore R_{\mu\nu} = T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R$$

$$= \frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{4} g_{\mu\nu}(\partial\phi)^2$$

$$- \frac{\epsilon e^{\beta\phi}}{4m!} g_{\mu\nu} F^2 + \frac{\epsilon e^{\beta\phi}}{2(m-1)!} F_{\mu a_2 \ldots a_m} F_\nu^{\;a_2\ldots}$$

$$- \frac{1}{2} g_{\mu\nu} \frac{1}{d-2} g^{\mu\nu} \partial_\mu\phi\,\partial_\nu\phi$$

$$+ \frac{1}{4(d-2)} g_{\mu\nu}\, d\,(\partial\phi)^2 + \frac{\epsilon e^{\beta\phi} d\, g_{\mu\nu}}{4m!\,(d-2)} F^2$$

$$- \frac{\epsilon e^{\beta\phi} m\, g_{\mu\nu}}{2m!\,(d-2)} F^2$$

$$= \frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2(d-2)}\, d\,\partial_\mu\phi\,\partial_\nu\phi$$

$$- \frac{1}{4} g_{\mu\nu}(\partial\phi)^2 + \frac{1}{4(d-2)} g_{\mu\nu}\, d\,(\partial\phi)^2$$

$$- \frac{\epsilon e^{\beta\phi}}{4m!} g_{\mu\nu} F^2 + \frac{\epsilon e^{\beta\phi}}{2(m-1)!} F_{\mu a_2 \ldots a_m} F_\nu^{\;a_2 \ldots a_m}$$

$$+ \frac{\epsilon e^{\beta\phi} d\, g_{\mu\nu}}{4m!\,(d-2)} - \frac{\epsilon e^{\beta\phi} m\, g_{\mu\nu}}{2m!\,(d-2)}$$

$$= \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} g_{\mu\nu} (\partial\phi)^2$$

$$- \frac{1}{2(d-2)} g_{\mu\nu} (\partial\phi)^2 + \frac{1}{4(d-2)} g_{\mu\nu} d (\partial\phi)^2$$

$$- \frac{\epsilon e^{\beta\phi}}{4m!} g_{\mu\nu} F^2 + \frac{\epsilon e^{\beta\phi}}{2(m-1)!} F_{\mu a_2 \dots a_m} F_\nu{}^{a_2 \dots a_m}$$

$$+ \frac{\epsilon e^{\beta\phi} d\, g_{\mu\nu}}{4m!(d-2)} F^2 - \frac{\epsilon e^{\beta\phi} m\, g_{\mu\nu} F^2}{2m!(d-2)}$$

$$= \frac{1}{2} \partial_\mu \phi \partial_\nu \phi$$

$$+ \frac{\epsilon e^{\beta\phi}}{2(m-1)!} \Big[F_{\mu a_2 \dots a_m} F_\nu{}^{a_2 \dots a_m}$$

$$- \frac{m-1}{m(d-2)} F_m^2 \, g_{\mu\nu} \Big]$$

$$\therefore R_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{\epsilon e^{\beta\phi}}{2(m-1)!}$$

$$[F_{\mu a_2 \dots a_m} F_\nu{}^{a_2 \dots a_m} - \frac{m-1}{m(d-2)} F_m^2 g_{\mu\nu}$$

Varying with scalar.

$$\delta S_\phi = \int d^d x \Big[\sqrt{|g|}\, \nabla_\mu \left(g^{\mu\nu} \partial_\mu \phi \right)$$

$$- \sqrt{|g|}\, \frac{\epsilon}{2m!}\, \beta\, e^{\beta\phi}\, F_m^2 \Big]\, \delta\phi$$

$$\delta S_\phi = 0$$

$$\cancel{\partial_\mu \sqrt{|g|}\, e} \;\therefore\; \partial_\mu \sqrt{|g|}\, \partial^\mu \phi$$

$$= \sqrt{|g|}\, \frac{\epsilon}{2m!}\, \beta\, e^{\beta\phi}\, F_m^2$$

$$\therefore \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|}\, \partial^\mu \phi \right) = \frac{\beta}{2m!} \epsilon\, e^{\beta\phi} F_m^2$$

varying with potential we get

$$\partial_\mu \left(\sqrt{|g|}\, e^{\beta\phi} F^{\mu\nu_2 \dots \nu_m} \right) = 0$$

Solve from equation of motion

$$* \; S = \int d^d \sqrt{|g|} \left(R - \frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{\epsilon}{2m!} e^{\beta\phi} F_m^2 \right)$$

$$dS^2 = e^{2U(\tau)} \left(\epsilon_0 \, d\tau^2 + \sum_{i=1}^{p} \epsilon_i \, \tau^{2a_i} \, dx_i^2 \right) + e^{2V(\tau)} \left(\sum_{j=p+1}^{d-1} \epsilon_j \, \tau^{2a_j} \, dx_j^2 \right)$$

here, $g_{\tau\tau} = \epsilon_0 \, e^{2U(\tau)}$

$$g_{ii} = \epsilon_i \, e^{2U(\tau)} \tau^{2a_i}$$

$$g_{jj} = \epsilon_j \, e^{2V(\tau)} \tau^{2a_j}$$

$$g^{\tau\tau} = \frac{e^{-2U(\tau)}}{\epsilon_0}$$

$$g^{ii} = \frac{e^{-2U(\tau)} \tau^{-2a_i}}{\epsilon_i}$$

$$g^{jj} = \frac{e^{-2V(\tau)} \tau^{-2a_j}}{\epsilon_j}$$

Christoffel symbols :

$$\Gamma^{\tau}_{\tau\tau} = \frac{1}{2} g^{\tau\tau} (\partial_\tau g_{\tau\tau})$$

$$= \frac{1}{2} \frac{e^{-2U}}{\epsilon_0} 2 e^{2U} \epsilon_0 \dot{U} = \dot{U}$$

$$\Gamma^{\tau}_{ii} = \frac{1}{2} g^{\tau\tau} (-\partial_\tau g_{ii})$$

$$= \frac{-1}{2} \frac{e^{-2U}}{\epsilon_0} \partial_\tau (\epsilon_i e^{2U} \tau^{2a_i})$$

$$= \frac{-\epsilon_i \tau^{2a_i}}{\epsilon_0} \left(\dot{U} + \frac{a_i}{\tau}\right), \quad i = 1 \dots p$$

$$\Gamma^{\tau}_{\partial\partial} = \frac{1}{2} g^{\tau\tau} (-\partial_\tau g_{\partial\partial})$$

$$= -\frac{1}{2} \frac{e^{-2U}}{\epsilon_0} \partial_\tau (\epsilon_{\partial} e^{2V} \tau^{2a_{\partial}})$$

$$= \frac{-\epsilon_{\partial} e^{2(V-U)}}{\epsilon_0} \tau^{2a_{\partial}} \left(\dot{V} + \frac{a_{\partial}}{\tau}\right)$$

$$\Gamma^{i}_{\tau i} = \Gamma^{i}_{i\tau} = \frac{1}{2} g^{ii} \partial_\tau g_{ii}$$

$$= \frac{1}{2} e^{-2U} \tau^{-2a_i} 2 e^{2U} \tau^{2a_i} \left(\dot{U} + \frac{a_i}{\tau}\right)$$

$= \dot{U} + \frac{a_i}{\tau}, \quad i = 1 \ldots p$

$\Gamma^{j}_{\tau j} = \Gamma^{j}_{j\tau} = \dot{V} + \frac{a_j}{\tau}, \quad j = p+1 \ldots d-1$

$R_{\mu\nu} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\lambda\sigma}\Gamma^{\sigma}_{\mu\nu}$

$- \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\lambda}$

$R_{\tau\tau} = \cancel{\partial_\tau \Gamma^{\tau}_{\tau\tau}} - \sum_{i=1}^{p} \partial_\tau \Gamma^{i}_{\tau i} - \sum_{j=p+1}^{d-1} \partial_\tau \Gamma^{j}_{\tau j} - \partial_\tau \Gamma^{\tau}_{\tau\tau}$

$+ \Gamma^{\lambda}_{\lambda\tau}\Gamma^{\tau}_{\tau\tau} - \Gamma^{\lambda}_{\tau\sigma}\Gamma^{\sigma}_{\tau\lambda}$

Now, $\Gamma^{\lambda}_{\lambda\tau}\Gamma^{\tau}_{\tau\tau}$

$= \left[\Gamma^{\tau}_{\tau\tau} + \sum \Gamma^{i}_{i\tau} + \sum \Gamma^{j}_{j\tau} \right] \Gamma^{\tau}_{\tau\tau}$

$= \left(\dot{U} + \sum \dot{U} + \frac{a_i}{\tau} + \sum \dot{V} + \frac{a_j}{\tau} \right) \dot{U}$

$$\Gamma^{\lambda}_{\tau\sigma}\,\Gamma^{\sigma}_{\tau\lambda}$$

$$=\sum\Gamma^{i}_{\tau i}\,\Gamma^{i}_{\tau i} + \sum\Gamma^{j}_{\tau j}\,\Gamma^{j}_{\tau j}$$

$$=\sum\left(\dot U+\frac{a_i}{\tau}\right)^2+\sum\left(\dot V+\frac{a_j}{\tau}\right)^2$$

$\therefore R_{\tau\tau}$

$$= \ddot U - \sum_{i=1}^{p}\ddot U - \frac{a_i}{\tau^2} - \sum_{j=p+1}^{d-1}\ddot V - \frac{a_j}{\tau^2} -$$

$$-\ddot U + \dot U^2 + \dot U\sum_{i=1}^{p}\left(\dot U+\frac{a_i}{\tau}\right)$$

$$+\dot U\sum_{j=p+1}^{d-1}\left(\dot V+\frac{a_j}{\tau}\right) - \sum_{i=1}^{p}\left(\dot U+\frac{a_i}{\tau}\right)^2$$

$$-\sum_{j=p+1}^{d-1}\left(\dot V+\frac{a_j}{\tau}\right)^2$$

R_{ii}

$$= -p\ddot U - q\ddot V + \frac{\sum a_i}{\tau^2} + \frac{\sum a_j}{\tau^2} + \ddot U$$

$$-\ddot{U} + \dot{U}^2 + p\dot{U}\dot{U} + \dot{U}\frac{\Sigma a_i}{\tau}$$

$$+ \dot{U} q \dot{V} + \dot{U}\frac{\Sigma a_j}{\tau}) - p\dot{U}^2 - \frac{2\dot{U}\Sigma a_i}{\tau}$$

$$- \frac{(\Sigma a_i)^2}{\tau^2} - q\dot{V}^2 - \frac{2\dot{V}\Sigma a_j}{\tau} - \frac{(\Sigma a_j)^2}{\tau^2}$$

$$= -p\ddot{U} - q\ddot{V} + \frac{S}{\tau^2} + \frac{l}{\tau^2}$$

$$-\ddot{V} + \dot{U}^2 + p\dot{U}^2 + \frac{\dot{U}S}{\tau} + \dot{U}q\dot{V}$$

$$+ \dot{U}\frac{l}{\tau} - p\dot{U}^2 - \frac{2\dot{U}S}{\tau} - \frac{(\Sigma a_i)^2}{\tau^2}$$

$$- q\dot{V}^2 - \frac{2\dot{V}l}{\tau} - \frac{(\Sigma a_j)^2}{\tau^2}$$

$$= -q\ddot{V} - p\ddot{U} - q\dot{V}(\dot{V} - \dot{U})$$

$$- \frac{1}{\tau}((S-l)\dot{U} + 2l\dot{V}) + \frac{1}{\tau^2}\sum_{k=1}^{d-1}(a_k^2 - a_k)$$

(2) $R_{x^i x^i}$

$$R_{ii} = \partial_\tau \Gamma^\tau_{ii} + \Gamma^\tau_{ii}\left(\Gamma^\tau_{\tau\tau} + \sum_k \Gamma^k_{\tau k}\right)$$

$$- 2\Gamma^i_{i\tau}\Gamma^\tau_{ii}$$

$$\partial_\tau\left[-\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i}\left(\dot{U} + \frac{a_i}{\tau}\right)\right]$$

$$= \partial_\tau\left[-\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i}\dot{U} - \frac{\epsilon_i}{\epsilon_0}\tau^{2a_i - 1}a_i\right] -$$

$$= -\frac{\epsilon_i}{\epsilon_0}2a_i\tau^{2a_i - 1}\dot{U} - \frac{\epsilon_i}{\epsilon_0}\tau^{2a_i}\ddot{U} - =$$

$$-\frac{\epsilon_i}{\epsilon_0}(2a_i - 1)\tau^{2a_i - 2}a_i$$

$$\Gamma^{\tau}_{ii}\Gamma^{\tau}_{\tau\tau} + \Gamma^{\tau}_{ii}\sum_{K}\Gamma^{K}_{\tau K}$$

$$= -\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i}\left(\dot{U}+\frac{a_i}{\tau}\right)\dot{U}$$

$$+\left[-\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i}\left(\dot{U}+\frac{a_i}{\tau}\right)\right]$$

$$\left[\sum_{i=1}^{p}\Gamma^{i}_{\tau i} + \sum_{j=p+1}^{d-1}\Gamma^{j}_{\tau j}\right]$$

$$= -\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i}\left(\dot{U}+\frac{a_i}{\tau}\right)\dot{U}$$

$$+\left[-\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i}\left(\dot{U}+\frac{a_i}{\tau}\right)\right]$$

$$\left[\cancel{\sum_{i=1}^{p}} p\dot{U}+\frac{s}{\tau}+ q\dot{V}+\frac{l}{\tau}\right]$$

$$= -\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i}\dot{U}^2 - \frac{\epsilon_i}{\epsilon_0}\tau^{2a_i-1}a_i\dot{U}$$

$$+\left[-\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i}\dot{U}-\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i-1}a_i\right]$$

$$\left[p\dot{U}+q\dot{V}+\frac{s+l}{\tau}\right]$$

$$=-\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i}\dot{U}^2-\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i-1}a_i\dot{U}$$

$$-\frac{\epsilon_i}{\epsilon_0}p\tau^{2a_i}\dot{U}^2-\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i}q\dot{V}\dot{U}$$

$$-\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i}\dot{U}\frac{s+l}{\tau}-\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i-1}a_i p\dot{U}$$

$$-\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i-1}a_i q\dot{V}-\frac{\epsilon_i}{\epsilon_0}\tau^{2a_i-1}a_i\frac{s+l}{\tau}$$

$-$

$\therefore R_{ii}$

$$= -\frac{\epsilon_i}{\epsilon_0} 2a_i \tau^{2a_i - 1} \dot{U} - \frac{\epsilon_i}{\epsilon_0} \tau^{2a_i} \ddot{U}$$

$$- \frac{\epsilon_i}{\epsilon_0} (2a_i - 1) \tau^{2a_i - 2} a_i - \frac{\epsilon_i}{\epsilon_0} \tau^{2a_i} \dot{U}^2$$

$$- \frac{\epsilon_i}{\epsilon_0} \tau^{2a_i - 1} a_i \dot{U} - \frac{\epsilon_i}{\epsilon_0} p \tau^{2a_i} \dot{U}^2 - \frac{\epsilon_i}{\epsilon_0} \tau^{2a_i} q \dot{V} \dot{U}$$

$$- \frac{\epsilon_i}{\epsilon_0} \tau^{2a_i} \dot{U} \frac{s+l}{\tau} - \frac{\epsilon_i}{\epsilon_0} \tau^{2a_i - 1} a_i p \dot{U}$$

$$- \frac{\epsilon_i}{\epsilon_0} \tau^{2a_i - 1} a_i q \dot{V} - \frac{\epsilon_i}{\epsilon_0} \tau^{2a_i - 1} a_i \frac{s+l}{\tau}$$

$$+ 2\left(\dot{U} + \frac{a_i}{\tau}\right) \frac{\epsilon_i}{\epsilon_0} \tau^{2a_i} \left(\dot{U} + \frac{a_i}{\tau}\right)$$

$$= -\epsilon_0 \epsilon_i 2a_i \tau^{2a_i - 1} \dot{U} - \epsilon_i \epsilon_0 \tau^{2a_i} \ddot{U}$$

$$- 2\epsilon_0 \epsilon_i a_i^2 \tau^{2a_i - 2} + \epsilon_0 \epsilon_i a_i \tau^{2a_i - 2}$$

$$- \epsilon_0 \epsilon_i \tau^{2a_i} \dot{U}^2 - \epsilon_0 \epsilon_i \tau^{2a_i - 1} a_i \dot{U}.$$

$$- \epsilon_0 \epsilon_i p \tau^{2a_i} \dot{U}^2 - \epsilon_0 \epsilon_i \tau^{2a_i} q \dot{U} \dot{V} - \epsilon_0 \epsilon_i \tau^{2a_i} \dot{U} \frac{s+l}{\tau}$$

$$-\epsilon_0\epsilon_i \tau^{2a_i-1} a_i p\dot{U} - \epsilon_0\epsilon_i \tau^{2a_i-1} a_i q\dot{V}$$

$$-\epsilon_0\epsilon_i \tau^{2a_i-1} a_i \frac{s+l}{\tau} + 2\epsilon_i\epsilon_0 \tau^{2a_i} \dot{U}^2$$

$$+ 2\epsilon_i\epsilon_0 \tau^{2a_i-1} \dot{U} a_i + 2\epsilon_i\epsilon_0 a_i \dot{U} \tau^{2a_i-1}$$

$$+ 2\epsilon_i\epsilon_0 a_i^2 \tau^{2a_i-2}$$

$$= -\epsilon_i\epsilon_0 \tau^{2a_i}\ddot{U} + \epsilon_0\epsilon_i a_i \tau^{2a_i-2}$$

$$+ \epsilon_0\epsilon_i \tau^{2a_i}\dot{U}^2 + \epsilon_0\epsilon_i \tau^{2a_i-1} a_i \dot{U}$$

$$- \epsilon_0\epsilon_i p \tau^{2a_i}\dot{U}^2 - \epsilon_0\epsilon_i \tau^{2a_i} q\dot{U}\dot{V}$$

$$- \epsilon_0\epsilon_i \tau^{2a_i} \dot{U} \frac{s+l}{\tau} - \epsilon_0\epsilon_i \tau^{2a_i-1} a_i p\dot{U}$$

$$- \epsilon_0\epsilon_i \tau^{2a_i-1} a_i q\dot{V} - \epsilon_0\epsilon_i \tau^{2a_i-1} a_i \frac{s+l}{\tau}$$

$$= -\epsilon_0\epsilon_i \tau^{2a_i}\Big[\ddot{U} - \frac{a_i}{\tau^2}$$

$$+ \left(\dot{U} + \frac{a_i}{\tau}\right)\left((p-1)\dot{U} + q\dot{V} + \frac{l+s}{\tau}\right)\Big]$$

$R_{x^j x^j}$

$$= \partial_\tau \Gamma^\tau_{\dot{j}\dot{j}} + \Gamma^\tau_{\dot{j}\dot{j}} \left(\Gamma^\tau_{\tau\tau} + \sum_K \Gamma^K_{\tau K} \right)$$

$$- 2 \Gamma^{j}_{\dot{j}\tau} \Gamma^\tau_{\dot{j}\dot{j}}$$

$$\partial_\tau \Gamma^\tau_{\dot{j}\dot{j}} = \partial_\tau \left[- \frac{\epsilon_j}{\epsilon_0} e^{2(v-u)} \tau^{2a_j} \left(\dot{v} + \frac{a_j}{\tau} \right) \right]$$

$$= \partial_\tau \left[- \frac{\epsilon_j}{\epsilon_0} e^{2(v-u)} \tau^{2a_j} \dot{v} - \frac{\epsilon_j}{\epsilon_0} e^{2(v-u)} a_j \tau^{2a_j - 1} \right]$$

$$= - \frac{\epsilon_j}{\epsilon_0} e^{2(v-u)} \tau^{2a_j} \left[2(\dot{v} - \dot{u}) \left(\dot{v} + \frac{a_j}{\tau} \right) + \frac{2a_j}{\tau} \left(\dot{v} + \frac{a_j}{\tau} \right) + \ddot{v} - \frac{a_j}{\tau^2} \right]$$

$$\Gamma^{r}_{jj}\,\Gamma^{r}_{rr} \;+\; \Gamma^{r}_{jj}\sum_{k}\Gamma^{k}_{rk}$$

$$= -\frac{\epsilon_j}{\epsilon_0}\, e^{2(V-U)}\, r^{2a_j}\,\dot{U}\left(\dot{V} + \frac{a_j}{r}\right)$$

$$+ -\frac{\epsilon_j}{\epsilon_0}\, e^{2(V-U)}\, r^{2a_j}\left(\dot{V} + \frac{a_j}{r}\right)$$

$$\left[\Gamma^{i}_{ri} + \Gamma^{j}_{rj} + \cancel{\Gamma^{r}_{rr}}\right]$$

$$= -\frac{\epsilon_j}{\epsilon_0}\, e^{2(V-U)}\, r^{2a_j}\,\dot{U}\dot{V}$$

$$- \frac{\epsilon_j}{\epsilon_0}\, e^{2(V-U)}\, r^{2a_j-1}\, a_j\,\dot{U} +$$

$$\left(-\frac{\epsilon_j}{\epsilon_0}\, e^{2(V-U)}\, r^{2a_j}\,\dot{V} - \frac{\epsilon_j}{\epsilon_0}\, e^{2(V-U)}\, a_j\, r^{2a_j-1}\right)$$

$$\left(\dot{U} + \frac{a_i}{r} + \dot{V} + \frac{a_j}{r}\right)$$

$$= -\frac{\epsilon_j}{\epsilon_0} e^{2(V-U)} r^{2a_j} \dot{U}\dot{V}$$

$$-\frac{\epsilon_j}{\epsilon_0} e^{2(V-U)} r^{2a_j-1} a_j \dot{U} +$$

$$-\frac{\epsilon_j}{\epsilon_0} e^{2(V-U)} r^{2a_j} \dot{U}\dot{V} - \frac{\epsilon_j}{\epsilon_0} e^{2(V-U)} r^{2a_j-1} a_i \dot{V} \Big] - \frac{\epsilon_j}{\epsilon_0} e^{2(V-U)} \dot{V}^2 r^{2a_j} =$$

$$-\frac{\epsilon_j}{\epsilon_0} e^{2(V-U)} r^{2a_j-1} a_j \dot{V} +$$

$$-\frac{\epsilon_j}{\epsilon_0} e^{2(V-U)} a_j r^{2a_j-1} \dot{U} +$$

$$-\frac{\epsilon_j}{\epsilon_0} e^{2(V-U)} a_i a_j r^{2a_j-2} +$$

$$-\frac{\epsilon_j}{\epsilon_0} e^{2(V-U)} a_j \dot{V} r^{2a_j-1}$$

$$-\frac{\epsilon_j}{\epsilon_0} e^{2(V-U)} a_j^2 r^{2a_j-2}$$

$R_{x^j x^j}$

$= \partial_\tau \Gamma^{\tau}_{jj} + \Gamma^{\tau}_{jj}\left(\Gamma^{\tau}_{\tau\tau} + \sum_K \Gamma_{\tau K}^{K}\right)$

$- 2\Gamma^{j}_{j\tau}\Gamma^{\tau}_{jj}$

$= -\frac{\epsilon_j}{\epsilon_0} e^{2(V-U)} \tau^{2a_j} \left[2\dot{V}^2\right.$

$+ 2\frac{\dot{V} a_j}{\tau} - 2\dot{U}\dot{V} - \frac{2\dot{U} a_j}{\tau}$

$\left. + \frac{2a_j \dot{V}}{\tau} + \frac{2a_j^2}{\tau^2} + \ddot{V} - \frac{a_j}{\tau^2}\right]$

$+ \left(-\frac{\epsilon_j}{\epsilon_0} e^{2(V-U)} \tau^{2a_j} \dot{U}\dot{V} - \frac{\epsilon_j}{\epsilon_0}\right.$

$\left.\dot{U} e^{2(V-U)} \tau^{2a_j-1} a_j\right)$

$\left(-\frac{\epsilon_j}{\epsilon_0} e^{2(V-U)} \tau^{2a_j} \dot{U}\dot{V}\right.$

$- \frac{\epsilon_j}{\epsilon_0} e^{2(V-U)} \tau^{2a_j-1} a_i \dot{V} - \frac{\epsilon_j}{\epsilon_0} e^{2(V-U} \dot{V}^2 \tau^{2a_j}$

$$- \frac{\epsilon_j}{\epsilon_0} e^{2(v-u)} \tau^{2a_j - 1} a_j \dot{v}$$

$$- \frac{\epsilon_j}{\epsilon_0} e^{2(v-u)} a_j \tau^{2a_j - 1} \dot{u}$$

$$- \frac{\epsilon_j}{\epsilon_0} e^{2(v-u)} a_i a_j \tau^{2a_j - 2}$$

$$- \frac{\epsilon_j}{\epsilon_0} e^{2(v-u)} a_j \dot{v} \tau^{2a_j - 1}$$

$$- \frac{\epsilon_j}{\epsilon_0} e^{2(v-u)} a_j^2 \tau^{2a_j - 2}$$

$$+ 2\left(\dot{v} + \frac{a_j}{\tau}\right) \frac{\epsilon_j}{\epsilon_0} e^{2(v-u)} \tau^{2a_j} \left(\dot{v} + \frac{a_j}{\tau}\right)$$

$$= -\epsilon_0 \epsilon_j e^{2(v-u)} \tau^{2a_j} \left[\ddot{v} - \frac{a_j}{\tau^2} + \left(\dot{v} + \frac{a_j}{\tau}\right)\left(q\dot{v} + (p-1)\dot{u} + \frac{a_s}{\tau}\right) \right]$$

SO.

$$R_{\tau\tau} = -q\ddot{V} - p\ddot{U} - q\dot{V}(\dot{V}-\dot{U}) - \frac{1}{\tau}((S-1)\dot{U} + 2l\dot{V}) - \frac{1}{\tau^2}\sum_{K=1}^{d-1}(a_K^2 - a_K)$$

$$R_{x_ix_i} = -\epsilon_0\epsilon_i \tau^{2a_i}\left[\ddot{U} - \frac{a_i}{\tau^2} + \left(\dot{U} + \frac{a_i}{\tau}\right)\left((p-1)\dot{U} + q\dot{V} + \frac{l+S}{\tau}\right)\right]$$

$$R_{x_jx_j} = -\epsilon_0\epsilon_j e^{2(V-U)}\tau^{2a_j}\left[\ddot{V} - \frac{a_j}{\tau^2} + \left(\dot{V} + \frac{a_j}{\tau}\right)\left(q\dot{V} + (p-1)\dot{U} + \frac{l+S}{\tau}\right)\right]$$

We impose relations.

$$\cancel{l = \sum} \quad \cancel{l = \sum_{j-1} a_j}$$

$$\sum_{K=1}^{d-1} a_K = \sum_{K=1}^{d-1} a_K^2 = 1$$

and $aV + (p-1)U = 0$

or, $aV = -(p-1)U$

$$\therefore V = -\frac{(p-1)U}{a}$$

$$\therefore \dot{V} = \frac{-(p-1)\dot{U}}{a}$$

$$\ddot{V} = \frac{-(p-1)\ddot{U}}{a}$$

$$\therefore R_{rr} = -a\left[-\frac{(p-1)\ddot{U}}{a}\right] - p\ddot{U}$$

$$-a\left[-\frac{(p-1)\dot{U}}{a}\right]\left[-\frac{(p-1)\dot{U}}{a} - \dot{U}\right]$$

$$-\frac{1}{r}\left((s-l)\dot{U} + 2l\left(-\frac{(p-1)\dot{U}}{a}\right)\right)$$

$$= \cancel{b}\ p\ddot{U} - \ddot{U} \cancel{p\ddot{U}} - p\ddot{U} - \frac{p\dot{U}^2(p-1)}{a}$$

$$-p\dot{U}^2 + \frac{(p-1)\dot{U}^2}{a} + \dot{U}^2 - \frac{1}{r}s\dot{U} + \frac{1}{r}l\dot{U}$$

$$+ \frac{2l}{r q}(p-1)\ddot{U}$$

$$= -\ddot{U} - \left[1 - 2\left(\frac{d-2}{q}\right)l\right]\frac{\dot{U}}{r}$$

$$- \frac{(p-1)(d-2)}{q}\dot{U}^2$$

[using the conditions $s+l = 1$, & $d = p+q+1$]

Now, from the first equation,

similarly.

$$R_{x_i x_i} = -\epsilon_0 \epsilon_i r^{-2a_i}\left(\ddot{U} + \frac{1}{r}\dot{U}\right)$$

$$R_{x_j x_j} = -\frac{(p-1)}{q}\epsilon_0 \epsilon_j e^{2\left(\frac{2-d}{q}\right)U} r^{2a_j}\left(\ddot{U} + \frac{\dot{U}}{r}\right).$$

We consider solutions with a p form given by

$$F_p = P\, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_p$$

The equation of motions are:

$$R_{uv} = \frac{1}{24} \partial_u \phi \partial_v \phi + \frac{\epsilon e^{\beta\phi}}{2(m-1)!} \Big[F_{u a_2 \ldots a_m} F_v{}^{a_2 \ldots a_m} - \frac{(p-1)}{p(d-2)} F_p^2 g_{uv} \Big]$$

Considering

$$F_p = \rho \, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_p$$

The first equation of motion

$$R_{\mu\nu} = \frac{1}{2} \partial_\mu \phi \, \partial_\nu \phi + \frac{\epsilon e^{\beta\phi}}{2(m-1)!}$$

$$\left[F_{\mu \partial_2 \ldots \partial_m} F_\nu^{\partial_2 \ldots \partial_m} - \frac{(p-1)}{p(d-2)} F_p^2 g_{\mu\nu} \right]$$

$$g^{ii} = \frac{e^{-2U\frac{q}{?}-2a_i}}{\epsilon_i}$$

$$F_p^2 = \frac{1}{p!} F_{\mu_1 \ldots \mu_p} F^{\mu_1 \ldots \mu_p}$$

$$F_{\mu_1 \ldots \mu_p} F^{\mu_1 \ldots \mu_p} = \frac{p! \, \rho^2 e^{-2pU} e^{-2S}}{\epsilon \, \epsilon_1 \ldots \epsilon_p}$$

$$\left[\because (1) F_{\mu_1 \ldots \mu_p} F^{\mu_1 \ldots \mu_p} = p! \, F_{1 \ldots p} F^{1 \ldots p}, \right.$$

(2) $F^{1 \ldots p} = \left(\prod_{i=1}^{p} g^{ii} \right) F_{1 \ldots p}$,

(3) $\left. F_{1 \ldots p} = \rho \right]$

So, we obtain.

$$F_p^2 = \frac{p!\, p^2 e^{-2pU} \tau^{-2S}}{\epsilon_1 \ldots \epsilon_p}$$

$$R_{\tau\tau} = \frac{1}{2}\dot{\phi}^2 + \frac{\epsilon e^{\beta\phi}}{2(p-1)!}\left(-\frac{(p-1)}{p(d-2)}\frac{p^2 e^{-2pU}\tau^{-2S}}{\epsilon_1 \ldots \epsilon_p} e^{2U}\epsilon_0\right)$$

or, $$-\ddot{U} - \left[1 - 2\left(\frac{d-2}{\alpha}\right)l\right]\frac{\dot{U}}{\tau}$$

$$- \frac{(p-1)(d-2)}{\alpha}\dot{U}^2 = \frac{1}{2}\dot{\phi}^2$$

$$- \frac{\epsilon\epsilon_0 e^{2(1-p)U + \beta\phi} p^2 \tau^{-2S}}{\epsilon_1 \ldots \epsilon_p \; p!} p! \frac{(p-1)}{d-2)}$$

$$\therefore \ddot{U} + \left[1 - 2\left(\frac{d-2}{\alpha}\right)l\right]\frac{\dot{U}}{\tau} =$$

$$+ \frac{(p-1)(d-2)}{\alpha}\dot{U}^2 = -\frac{\dot{\phi}^2}{2} + \frac{(p-1)}{2(d-2)}\, \partial$$

where, $\dot{J} = \epsilon \epsilon_0 \epsilon_1 \ldots \epsilon_p \, e^{2(1-p)U + \beta\phi} \, P^2 \, \tau^{-2S}$

Again from the first equation of motion

$$R_{ii} = \frac{\epsilon e^{\beta\phi}}{2(p-1)!} \Big[F_{i a_2 \ldots a_p} F_i^{\;a_2 \ldots a_m} - \frac{(p-1)}{p(d-2)} F_p^2 \, g_{ii} \Big]$$

$$F_{i a_2 \ldots a_p} F_i^{\;a_2 \ldots a_p}$$

$$= (p-1)! \, \frac{P^2 \prod_{k=1}^{p} g^{kk}}{g^{ii}}$$

$$= (p-1)! \, P^2 \times \frac{e^{-2pU} \tau^{-2S}}{e^{-2U} \tau^{-2a_i}}$$

$$= (p-1)! \, P^2 \times e^{-2(p-1)U} \tau^{-2S} \tau^{2a_i}$$

So,

$$R_{ii} = \frac{\epsilon e^{\beta\phi}}{2(p-1)!}\left[(p-1)!\,p^2 e^{2(1-p)U}\tau^{-2s}\right.$$

$$\left.\frac{\tau^{2a_i}\epsilon_i}{\epsilon_1\ldots\epsilon_p} - \frac{(p-1)\epsilon_i}{p(d-2)}\frac{p!\,p^2 e^{2(1-p)U}\tau^{-2s}\tau^{2a_i}}{\epsilon_1\ldots\epsilon_p}\right]$$

$$= \frac{\epsilon e^{\beta\phi+2(1-p)U}\tau^{-2s}\tau^{2a_i}\epsilon_i p^2}{2\,\epsilon_1\ldots\epsilon_p}$$

$$- \frac{\epsilon e^{\beta\phi+2(1-p)U}\tau^{-2s}\tau^{2a_i}\epsilon_i p^2(p-1)}{2(d-2)\,\epsilon_1\ldots\epsilon_p}$$

or, $-\epsilon_0\epsilon_i\tau^{2a_i}\left(\ddot{U}+\frac{\dot{U}}{\tau}\right)$

$$=\epsilon\epsilon_i\epsilon_1\ldots\epsilon_p\, e^{\beta\phi+2(1-p)U}\tau^{-2s}\tau^{2a_i}p^2$$

$$\left(\frac{1}{2}-\frac{p-1}{2(d-2)}\right)$$

$$\therefore \ddot{U} + \frac{\dot{U}}{\tau} = -\frac{\alpha}{2(d-2)} \dot{J} \quad [d = p+\alpha+1]$$

where, $\dot{J} = \epsilon \epsilon_0 \epsilon_1 \ldots \epsilon_p \, e^{2(1-p)U + p\phi} \, \rho^2 \tau^{-2s}$

Now -

$$|g| = |g_{\tau\tau}| \prod_{i=1}^{p} |g_{ii}| \prod_{j=p+1}^{d-1} |g_{jj}|$$

$$= e^{2U} \epsilon_0 \; e^{2pU} \epsilon_1 \ldots \epsilon_p \, \tau^{2s} \, e^{2\alpha V} \epsilon_{p+1} \ldots \epsilon_{d-1} \, \tau^{2l}$$

$$= e^{2(U + pU + \alpha V)} \, \tau^{2(s+l)} \epsilon_0 \epsilon_1 \ldots \epsilon_p \epsilon_{p+1} \ldots \epsilon_{d-1}$$

$$\therefore \sqrt{|g|} = e^{(U + pU + \alpha V)} \, \tau^{s+l} \, \epsilon_0 \epsilon_1 \epsilon_p \epsilon_{p+1} \ldots \epsilon_{d-1}$$

$$\partial^\tau \phi = g^{\tau\tau} \partial_\tau \phi = \frac{e^{-2U}}{\epsilon_0} \dot{\phi}$$

$$\partial_\tau\left(\sqrt{|g|}\,\partial^\tau\phi\right)$$

$$= \partial_\tau\Big[e^{(U+pU+qV)}\,\tau^{s+l}\,\epsilon_0\epsilon_1\ldots\epsilon_p\epsilon_{p+1}\ldots\epsilon_{d-1}$$

$$\frac{e^{-2U}}{\epsilon_0}\,\dot{\phi}\Big]$$

$$= \partial_\tau\Big[e^{-U+pU+qV}\,\tau^{s+l}\,\dot{\phi}\,\epsilon_1\ldots\epsilon_p\epsilon_{p+1}\ldots\epsilon_{d-1}$$

Now,

$$\frac{1}{\sqrt{|g|}}\,\partial_\tau\left(e^{-U+pU+qV}\,\tau^{s+l}\,\dot{\phi}\,\epsilon_1\ldots\epsilon_p\epsilon_{p+1}\ldots\epsilon_{d-1}\right.$$

$$= \frac{1}{e^{(U+pU+qV)}\,\tau^{s+l}\,\epsilon_0\epsilon_1\ldots\epsilon_p\epsilon_{p+1}\ldots\epsilon_{d-1}}$$

$$\times\,\epsilon_1\ldots\epsilon_p\epsilon_{p+1}\ldots\epsilon_{d-1}\,\times$$

$$\Big[e^{(p-1)U+qV}\,\tau^{s+l}\,\ddot{\phi} + \left[(p-1)\dot{U}+q\dot{V}+\frac{s+l}{\tau}\right] e^{(p-1)U+qV}\,\tau^{s+l}\,\dot{\phi}\Big]$$

$$= \frac{e^{-2U}}{\epsilon_0}\ddot{\phi} + \left[(p-1)\dot{U} + q\dot{V} + \frac{s+l}{\tau}\right]\frac{e^{-2u}\dot{\phi}}{\epsilon_0}$$

We know the 3rd equation of mli-

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\,\partial^\mu\phi\right) = \frac{\beta}{2m!}\epsilon e^{\beta\phi}F_m^2$$

Now, here $\mu = \tau$.

$$\frac{e^{-2U}}{\epsilon_0}\ddot{\phi} + \left[(p-1)\dot{U} + q\dot{V} + \frac{s+l}{\tau}\right]\frac{e^{-2U}\dot{\phi}}{\epsilon_0}$$

$$= \frac{\beta}{2p!}\frac{\epsilon e^{\beta\phi}\, p!\, P^2 e^{-2pV}\tau^{-2s}}{\epsilon_1\cdots\epsilon_p}$$

or, $\ddot{\phi} + \left[(p-1)\dot{U} + q\dot{V} + \frac{s+l}{\tau}\right]\dot{\phi}$

$$= \frac{\beta}{2}\epsilon e^{\beta\phi + 2(1-p)U} P^2 \tau^{-2s} \epsilon_0\epsilon_1\cdots\epsilon_p$$

or, $\ddot{\phi} + \frac{1}{\tau}\dot{\phi} = \frac{\beta}{2}\partial$

So, we get.

$$\ddot{U} + \left[1 - 2\left(\frac{d-2}{\alpha}\right)l\right]\frac{\dot{U}}{\tau} + \frac{(p-1)(d-2)}{\alpha}\dot{U}^2 = -\frac{\dot{\phi}^2}{2} + \frac{(p-1)}{2(d-2)}\dot{J} \quad \text{--- (1)}$$

$$\ddot{U} + \frac{\dot{U}}{\tau} = -\frac{\alpha}{2(d-2)}\dot{J} \quad \text{--- (2)}$$

$$\frac{\dot{\phi}}{\tau} + \ddot{\phi} = \frac{\beta}{2}\dot{J} \quad \text{--- (3)}$$

From (2),

$$\dot{J} = -\frac{2(d-2)}{\alpha}\left(\ddot{U} + \frac{\dot{U}}{\tau}\right) \quad \text{--- (4)}$$

$$\therefore \frac{\dot{\phi}}{\tau} + \ddot{\phi} = \frac{\beta}{2}\left[-\frac{2(d-2)}{\alpha}\left(\ddot{U} + \frac{\dot{U}}{\tau}\right)\right]$$

$$\text{or, } \frac{\dot{\phi}}{\tau} + \ddot{\phi} = -\frac{\beta(d-2)}{\alpha}\left(\ddot{U} + \frac{\dot{U}}{\tau}\right)$$

$$\text{or, } \left(\ddot{\phi} + \frac{\dot{\phi}}{\tau}\right) + \frac{\beta(d-2)}{\alpha}\left(\ddot{U} + \frac{\dot{U}}{\tau}\right) = 0$$

or, $\left[\frac{d^2}{d\tau^2}+\frac{1}{\tau}\frac{d}{d\tau}\right]\left(\phi+\frac{\beta(d-2)}{2}u\right)=0$

So the combination

$$\psi(\tau)=\phi+\frac{\beta(d-2)}{2}u$$

satisfies the homogenous ODE.

$$\psi''+\frac{1}{\tau}\psi'=0$$

multiplying the equation by τ

$$\tau\psi''+\psi'=0$$

~~or, $\psi'=-\tau\psi''$~~

So, $\frac{d}{d\tau}(\tau\psi')=0$

$$\therefore \tau\psi'=c_1$$

$$\therefore \psi'=\frac{c_1}{\tau}$$

Integrating both sides with respect to τ.

$$\int \psi' dr = \int \frac{c_1}{r} dr$$

$$\therefore \psi = c_1 \ln r + c_2$$

or $\phi + \frac{\beta(d-2)}{\alpha} U = c_1 \ln r + c_2$

If we want the scalar fields to stay finite and not grow like $\ln r$ (which usually causes singularities), set $c_1 = 0$.

$$\therefore \phi + \frac{\beta(d-2)}{\alpha} U = c_2.$$

Now, $c_2 = 0$, as it verifies the previous equation.. $\left(\ddot{\phi} + \frac{\dot{\phi}}{r}\right)$

$$+ \beta \frac{(d-2)}{\alpha}\left(\ddot{U} + \frac{\dot{U}}{r}\right) = 0$$

$$\therefore \quad \phi = -\frac{\beta(d-2)}{\alpha} U$$

substituting $\phi = -\frac{\beta(d-2)}{\alpha} U$ into the equations (1),

$$\ddot{U} + \left[1 - 2\left(\frac{d-2}{\alpha}\right) l\right]\frac{\dot{U}}{r} + \frac{(p-1)(d-2)}{\alpha}\dot{U}^2$$

$$= -\frac{\beta^2 (d-2)^2}{2\alpha^2}\dot{U}^2 + \frac{(p-1)}{2(d-2)}\dot{J} \quad \text{———} \quad (4)$$

~~or, $\ddot{U} + [$~~ Now, from (2)

$$\ddot{U} + \frac{\dot{U}}{r} = -\frac{\alpha}{2(d-2)}\dot{J}$$

$$\text{or,} \quad \dot{J} = -\frac{2(d-2)}{\alpha}\left(\ddot{U} + \frac{\dot{U}}{r}\right) \quad \text{———} \quad (5)$$

putting (5) in (4).

$$\ddot{U} + \left[1 - 2\left(\frac{d-2}{\alpha}\right) l\right]\frac{\dot{U}}{r} + \frac{(p-1)(d-2)}{\alpha}\dot{U}^2$$

$$= -\frac{\beta^2}{2}\frac{(d-2)^2}{\alpha^2}\dot{U}^2 \cancel{+} - \frac{(p-1)}{2(d-2)}\,\frac{2(d-2)}{\alpha}\left(\ddot{U} + \frac{\dot{U}}{r}\right)$$

or,

$$\ddot{U} + \left(1 - 2\left(\frac{d-2}{a}\right)l\right)\frac{\dot{U}}{r} +$$

$$(p-1)\frac{(d-2)}{a}\dot{U}^2 = -\frac{1}{2}\beta^2\frac{(d-2)^2}{a^2}\dot{U}^2$$

$$-\frac{(p-1)}{a}\left(\ddot{U} + \frac{\dot{U}}{r}\right)$$

or, $$\ddot{U} + \left(1 - 2\frac{d-2}{a}l\right)\frac{\dot{U}}{r} +$$

$$\frac{(p-1)(d-2)}{a}\dot{U}^2 + \frac{1}{2}\cdot\beta^2\frac{(d-2)^2}{a^2}\dot{U}^2$$

$$+ \frac{(p-1)}{a}\left(\ddot{U} + \frac{\dot{U}}{r}\right) = 0$$

or, $$\left(1 + \frac{p-1}{a}\right)\ddot{U} +$$

$$\left[1 - 2\frac{d-2}{a}l + \frac{p-1}{a}\right]\frac{\dot{U}}{r} +$$

$$\left[\frac{(p-1)(d-2)}{a} + \frac{1}{2}\beta^2\frac{(d-2)^2}{a^2}\right]\dot{U}^2 = 0$$

Now dividing $\left(1+\frac{p-1}{a}\right)$ on both sides, we get,

$$\ddot{U}+\left[\frac{1-2\frac{d-2}{a}l+\frac{p-1}{a}}{1+\frac{p-1}{a}}\right]\frac{\dot{U}}{\tau}+$$

$$\left[\frac{\frac{(p-1)(d-2)}{a}+\frac{1}{2}\beta^2\frac{(d-2)^2}{a^2}}{1+\frac{p-1}{a}}\right]\dot{U}^2=0$$

Now,
$$1-2l=\frac{1-2\frac{d-2}{a}l+\frac{p-1}{a}}{1+\frac{p-1}{a}}$$

and,
$$\mu=\frac{(p-1)\frac{(d-2)}{a}+\frac{1}{2}\beta^2\frac{(d-2)^2}{a^2}}{1+\frac{p-1}{a}}$$

$$=\frac{a}{d-2}\left[\frac{(p-1)(d-2)}{a}+\frac{1}{2}\beta^2\frac{(d-2)^2}{a^2}\right]$$

$$=p-1+\frac{\beta^2(d-2)}{2a}$$

$$\therefore \ddot{U} + (1-2\theta)\frac{\dot{U}}{\tau} + \mu \dot{U}^2 = 0 \quad \text{———} \quad (6)$$

Again, from

$$\ddot{U} + \frac{\dot{U}}{\tau} = -\frac{\alpha}{2(d-2)}\ddot{\jmath}$$

we get,

$$\ddot{U} + \frac{\dot{U}}{\tau} + \frac{\alpha}{2(d-2)} \epsilon\epsilon_0\epsilon_1 \cdot \epsilon_p e^{2(1-p)U + \beta\phi} p^2 \tau^{-2s} = 0$$

or, $\ddot{U} + \frac{\dot{U}}{\tau} +$

$$\frac{\alpha}{2(d-2)} \epsilon\epsilon_0\epsilon_1 \cdots \epsilon_p e^{2(1-p)U + \beta\left(-\frac{\beta(d-2)}{\alpha}\right)U} p^2 \tau^{-2s}$$

or, $\ddot{U} + \frac{\dot{U}}{\tau} +$

$$\frac{\alpha}{2(d-2)} \epsilon\epsilon_0\epsilon_1 \cdots \epsilon_p e^{2(1-p)U - \frac{\beta^2(d-2)}{\alpha}U} p^2 \tau^{-2s} = 0$$

$$\therefore \ddot{U} + \frac{\dot{U}}{\tau} + \epsilon \epsilon_0 \epsilon_1 \cdots \epsilon_p \frac{\alpha e^{-2uU}}{2(d-2)} \rho^2 \tau^{-2s} = 0 \qquad \text{— (7)}$$

Now from (6),

$$\ddot{U} + (1-2l)\frac{\dot{U}}{\tau} + u\dot{U}^2 = 0$$

consider, $V = \dot{U}$

or, $\ddot{U} = \cancel{V}\ \dot{V}$

Then the equation becomes,

$$\dot{V} + (1-2l)\frac{V}{\tau} + uV^2 = 0$$

Now, consider, $\omega = V^{-1}$

$$\therefore V = \frac{1}{\omega}, \quad \dot{V} = -\frac{\dot{\omega}}{\omega^2}$$

Substituting $V = \frac{1}{\omega}$, $\dot{V} = -\frac{\dot{\omega}}{\omega^2}$.

$$-\frac{\dot{\omega}}{\omega^2} + (1-2l)\frac{1}{\tau}\frac{1}{\omega} + \frac{u}{\omega^2} = 0$$

Multiplying ω^2

$$-\dot{\omega} + (1-2l)\frac{\omega}{\tau} + u = 0$$

or, $$\dot{\omega} - (1-2l)\frac{\omega}{\tau} - u = 0$$

or, $$\frac{d\omega}{d\tau} - \frac{1-2l\ \omega}{\tau} = u$$

Now, the integrating factor,

$$= \exp\left(-\int \frac{1-2l}{\tau} d\tau\right)$$

$$= \tau^{-(1-2l)}$$

$$= \tau^{2l-1}$$

multiply the integrating factor on both sides.

$$\tau^{2l-1}\frac{d\omega}{d\tau} - (1-2l)\tau^{2l-2}\omega$$
$$= u\tau^{2l-1}$$

or, $$\frac{d}{d\tau}(\omega\tau^{2l-1}) = u\tau^{2l-1}$$

$$\therefore \omega\tau^{2l-1} = u\int \tau^{2l-1} d\tau + c_1$$

Or, $\omega r^{2l-1} = \mu . \frac{r^{2l}}{2l} + c_1$

$$\therefore \omega = \frac{\mu}{2l} r + \frac{c_1}{r^{2l-1}}$$

$$\therefore \dot{U} = \frac{1}{\omega} = \frac{1}{c_1 r^{1-2l} + \frac{\mu}{2l} r}$$

$$= \frac{1}{r^{1-2l}\left(c_1 + \frac{\mu}{2l} r^{2l}\right)}$$

$$= \frac{r^{2l-1}}{c_1 + c_2 r^{2l}}$$

Let, $X = c_1 + c_2 r^{2l}$

or, $dX = \frac{\mu}{2l} 2l \, r^{2l-1} dr$

$$\therefore \frac{dX}{\mu} = r^{2l-1} dr$$

Now, $\dot{U} = \frac{r^{2l-1}}{c_1 + c_2 r^{2l}}$

or, $\frac{dU}{dr} = \frac{r^{2l-1}}{c_1 + c_2 r^{2l}}$

or, $dU = \dfrac{r^{2l-1}\,dr}{c_1 + c_2 r^{2l}}$

or, $dU = \dfrac{dx/\mu}{x}$

or, $dU = \dfrac{1}{\mu}\dfrac{dx}{x}$

$\therefore\ U = \dfrac{1}{\mu}\ln x \pm \cancel{\text{constant}}\ c_3$

or, $U = \dfrac{1}{\mu}\ln\left(c_1 + c_2 r^{2l}\right) \cancel{\pm c_3}$

or, $U = \ln\left(c_1 + c_2 r^{2l}\right)^{\frac{1}{\mu}}$

$\therefore\ e^{\mu} = \left(c_1 + c_2 r^{2l}\right)^{\frac{1}{\mu}}$

8. Outlook:

The study of generalized kasner brane solutions as explored in the indernship provides a deeper understanding of higher dimensional gravitational theories and their interactions with antisymmetric tensor fields and scalar fields. By deriving the equations of motion from the Einstein Hilbert action coupled to m form fields and scalar fields and by obtaining explicit solutions for the function $e^{U(\tau)}$ and the scalar field $\phi(\tau)$, we have established a framework to analyze time dependent configurations. These solutions can offer a platform for further investigations into the dynamics of cosmological systems including singularity structure, stability and physical interpretations in the context of supergravity and string theory. Such works really could provide valuable insights into the dynamics of higher dimensional cosmological models.

Appendix:

(1) $\delta(\sqrt{|g|}R) = (\delta\sqrt{|g|})R + \sqrt{|g|}\,\delta R$

$\delta\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}\, g^{\mu\nu}\,\delta g_{\mu\nu}$

$= -\frac{1}{2}\sqrt{|g|}\, g_{\mu\nu}\,\delta g^{\mu\nu}$

Now, $\delta R = g^{\mu\nu}\,\delta R_{\mu\nu} + R_{\mu\nu}\,\delta g^{\mu\nu}$

Here, $\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda}$

So, $\delta R = R_{\mu\nu}\,\delta g^{\mu\nu}$

$+ g^{\mu\nu}(\nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda})$

$\therefore \delta(\sqrt{|g|}R)$

$= \sqrt{|g|}(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu})\,\delta g^{\mu\nu}$

$+ \sqrt{|g|}\,[\nabla_\lambda g^{\mu\nu}\,\delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu g^{\mu\nu}\,\delta\Gamma^\lambda_{\mu\lambda}]$

$= \sqrt{|g|}\,(R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu})\,\delta g_{\mu\nu} +$

$$+\boxed{\sqrt{|g|}\,\nabla_a\left(g^{\mu\nu}\,\delta\Gamma^{a}_{\mu\nu}-g^{\mu a}\,\delta\Gamma^{\nu}_{\mu\nu}\right)}$$

$\rightarrow$ boundary term

Here, $\int d^dx \sqrt{|g|}\,\nabla_a\left(g^{\mu\nu}\,\delta\Gamma^{a}_{\mu\nu}-g^{\mu a}\,\delta\Gamma^{\nu}_{\mu\nu}\right)$

$=0$

by considering the boundary terms vanishing