Quiver Representation: Gabriel's Theorem and Kac's Theorem

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§1 Quivers and their representations

A quiver is a directed graph Q, which we assume to be finite. A representation of a quiver Q over a field k is an assignment of

- a k-vector space V_i to each vertex i;
- and a k-linear map $V_\alpha : V_i \to V_j$ to each arrow $\alpha : i \to j$.

Formally speaking,

> **Definition 1.1** (Quiver)**.** A **quiver** $Q = (Q_0, Q_1, h, t)$ is a directed graph, where Q_0 is the (finite) set of vertices, Q_1 is the (finite) set of arrows, and $h, t : Q_1 \to Q_0$ are maps that take each edge to its head and tail, respectively.

> **Definition 1.2** (Quiver representation)**.** A **representation** V of a quiver $Q = (Q_0, Q_1, h, t)$ over a field k is an assignment of a k-vector space V_i to each vertex $i \in Q_0$, and a k-linear map $V_\alpha : V_{t(\alpha)} \to V_{h(\alpha)}$ to each arrow $\alpha \in Q_1$.

A representation V has a dimension vector $\underline{\dim} V$ which is the column vector of the dimensions of the V_i's for $i \in Q_0$. So $\underline{\dim} V \in \mathbb{N}^{Q_0} = \mathbb{N}^{|Q_0|}$,[1] once we fix an ordering $1, 2, \ldots, n$ of the vertices.

> **Definition 1.3.** Let V, W be representations of $Q = (Q_0, Q_1, h, t)$. A **morphism** $\phi : V \to W$ is an attachment $\phi_i : V_i \to W_i$ to each vertex $i \in Q_0$ such that the following diagram commutes for every $\alpha \in Q_1$:
>
> $$\begin{tikzcd} V_{t(\alpha)} \arrow[r, "V_\alpha"] \arrow[d, "\phi_{s(\alpha)}"'] & V_{h(\alpha)} \arrow[d, "\phi_{h(\alpha)}"] \\ W_{t(\alpha)} \arrow[r, "W_\alpha"'] & W_{h(\alpha)} \end{tikzcd}$$

The identity and compositions of morphisms can be defined naturally. Thus one can form the category $\operatorname{Rep} Q$ of representations of a quiver Q.

> **Definition 1.4.** Given a representation V of a quiver Q, a representation W is a **subrepresentation** of V if for each vertex $i \in Q_0$, W_i is a subspace of V_i and for each arrow $\alpha \in Q_1$,

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[1] $0 \in \mathbb{N}$.

> $W_\alpha : W_{t(\alpha)} \to W_{h(\alpha)}$ is a restriction of $V_\alpha : V_{t(\alpha)} \to V_{h(\alpha)}$.

Given any two representations V and W of a quiver Q, we can form the direct sum representation $V \oplus W$ by the natural construction.

> **Definition 1.5.** A representation U of a quiver Q is **indecomposable** if there does not exist nontrivial subrepresentations V and W such that $U \cong V \oplus W$. We denote by $\operatorname{Ind} Q$ the set of isomorphism classes of indecomposable representations of Q.

§2 Finite-type quivers

> **Definition 2.1.** A quiver Q is of **finite representation type** (or finite type, in short) if it only has finitely many isomorphism classes of indecomposable representations. In other words, $|\operatorname{Ind} Q| < \infty$.

Example 2.1. Consider the quiver

$$\bullet \longrightarrow \bullet$$

This quiver is known as A_2. A representation of this quiver is of the form

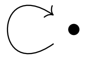
$$V \xrightarrow{\alpha} W.$$

Let V' is a complement of $\operatorname{Ker} \alpha \subseteq V$, and W' is a complement of $\operatorname{im} \alpha \subseteq W$. In other words, $V = \operatorname{Ker} \alpha \oplus V'$ and $W = W' \oplus \operatorname{im} \alpha$. Then we have

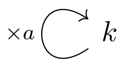
$$V \xrightarrow{\alpha} W = \quad \operatorname{Ker} \alpha \longrightarrow 0 \quad \oplus \quad V' \xrightarrow{\cong} \operatorname{im} \alpha \quad \oplus \quad 0 \longrightarrow W'. \tag{1}$$

The first summand can be decomposed into direct sum of a few copies of $k \to 0$; the second summand is a direct sum of some $k \xrightarrow{\mathbb{1}} k$; the last one is a direct sum of some $0 \to k$. Therefore, it has only 3 indecomposable representations, and their dimension vectors are $(1, 0), (1, 1), (0, 1)$.

Example 2.2. Now consider the quiver



We consider representations of the following form:



Two such representations (for $a \neq b$) are isomorphic if there exists an isomorphism $\phi : k \to k$ such that the following diagram commutes:

$$\begin{tikzcd}
k \arrow[r, "\times a"] \arrow[d, "\phi"'] & k \arrow[d, "\phi"] \\
k \arrow[r, "\times b"'] & k
\end{tikzcd}$$

This means for every $x \in k$,

$$\phi(ax) = b\phi(x). \tag{2}$$

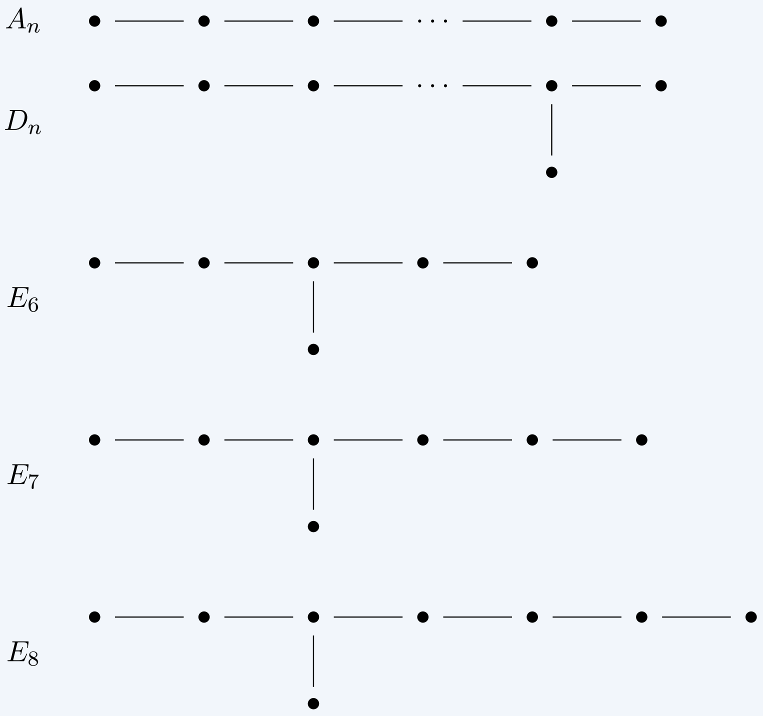
This is true if and only if $a = b$. Therefore, there pairwise non-isomorphic representations for each $a \in k$. Also, these are all indecomposable. When $|k|$ is infinite, there are infinitely many indecomposable representations.

In case of k being a finite field, we can just take the vector space k^n along with the endomorphism $J_n(\lambda)$, which is the $n \times n$ matrix with λ in the main diagonal, 1 in the superdiagonal, 0 elsewhere. This is an indecomposable representation. So we have at least one indecomposable representation for each dimension. In other words, we have infinitely many indecomposable representations. Hence, this graph is not of finite type.

Gabriel's theorem classifies all the finite type quivers.

Theorem 2.1 (Gabriel's theorem)

A connected quiver Q is of finite type if and only if the corresponding undirected graph (i.e. with directions of arrows forgotten) is one of the following:



These diagrams are called (simply laced) Dynkin diagrams.

Many classification problems feature these diagrams. Examples include graphs whose adjacency matrices have eigenvalues less than 2, finite type quivers, finite reflection groups, irreducible root systems, semisimple Lie algebras, among others. Collectively, these are known as the ADE classifications, named after the Dynkin diagrams A_n, D_n, E_6, E_7, E_8.

§3 Graphs of small eigenvalues

Let G be an undirected graph. Then we can consider its adjacency matrix A_G (labeling the vertices $1, 2, \ldots, n$):

$$[A_G]_{ij} = \text{ number of edges between } i \text{ and } j. \tag{3}$$

This is a real symmetric matrix. So all its eigenvalues are real. In this section, we shall classify all the graphs whose adjacency matrices have eigenvalues (strictly) less than 2.

Proposition 3.1

Let G' be a subgraph of G. Suppose $\lambda_{\max}$ and $\lambda'_{\max}$ are the highest eigenvalues of A_G and $A_{G'}$, respectively. Then $\lambda_{\max} \geq \lambda'_{\max}$.

Proof. Recall that the largest eigenvalue λ_A of an $n \times n$ real symmetric matrix A is given by

$$\lambda_A = \max_{\mathbf{x} \in \mathbb{R}^n, \|\mathbf{x}\|=1} \mathbf{x}^T A \mathbf{x}. \tag{4}$$

Let $\mathbf{w}$ be a unit eigenvector corresponding to the eigenvector $\lambda'_{\max}$ for $A_{G'}$. WLOG, the components of $\mathbf{w}$ are non-negative, since replacing w_i with $|w_i|$ only increases the quadratic form $\mathbf{x}^T A_{G'}\mathbf{x} = \sum_{i=1}^m x_i^2 \lambda_i$ (here, $m = |G'|$).

Now, we can assume A_G and $A_{G'}$ are of the same size (if some vertices are missing in G', we just put those entries to be 0s). Similarly, we can just extend $\mathbf{w}$ with 0s in the missing vertices. As a result,

$$\mathbf{w}^T A_{G'}\mathbf{w} \leq \mathbf{w}^T A_G \mathbf{w}, \tag{5}$$

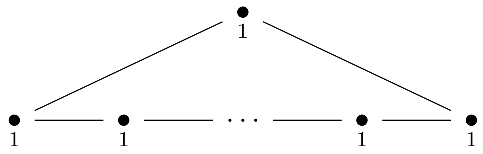
because adding 1 in the ij-th entry increases the value of the quadratic form by $w_i w_j$ (which is a non-negative number). Therefore,

$$\lambda'_{\max} = \mathbf{w}^T A_{G'}\mathbf{w} \leq \mathbf{w}^T A_G \mathbf{w} \leq \max_{\mathbf{x}\in\mathbb{R}^n, \|\mathbf{x}\|=1} \mathbf{x}^T A_G \mathbf{x} = \lambda_{\max}. \tag{6}$$

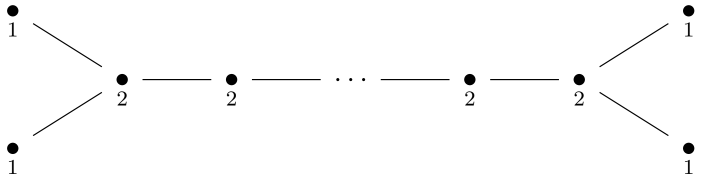
■

Now, consider the following graphs: (trust me, the labels are gonna make sense in a minute)

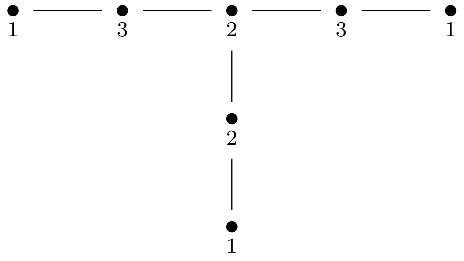
$\widetilde{A}_n$



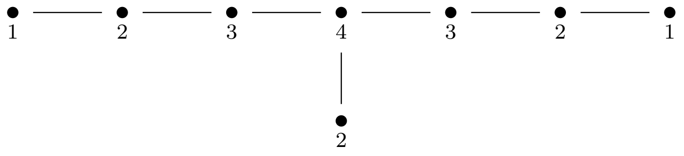
$\widetilde{D}_n$



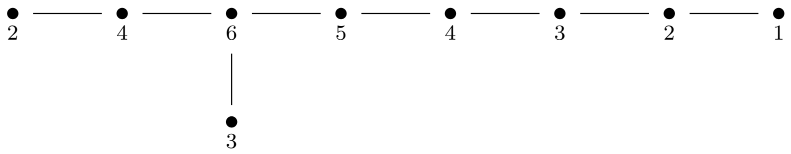
$\widetilde{E}_6$



$\widetilde{E}_7$



$\widetilde{E}_8$



Notice that twice the label on a vertex is equal the sum of the labels of its neighbors. Therefore, 2 is an eigenvalue for each of these graphs, because we can form a vector $\mathbf{v}$ with these labels, and then

$$[A\mathbf{v}]_j = \sum_{i=1}^n A_{ji} v_i = \sum_{\substack{i \text{ is a} \\ \text{neighbor of } j}} v_i = 2v_j, \tag{7}$$

so that $A\mathbf{v} = 2\mathbf{v}$. Therefore, if G is a graph with $\lambda_{\max} < 2$, then G cannot contain either of $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ as a subgraph.

Theorem 3.2

If G is a connected graph with $\lambda_{\max} < 2$, then G must be one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8.

Proof. We present the proof in the following steps.

1. Since G doesn't contain $\widetilde{A}_n$, it must be a tree (it automatically excludes self-loops or multi-edges).
2. Since G doesn't contain $\widetilde{D}_4$, it doesn't have any vertex with degree ≥ 4. So all the vertices have degree ≤ 3.
3. Since G doesn't contain $\widetilde{D}_n$ for $n \geq 5$, there is exactly one (if any) vertex with degree 3.
4. If none of the vertices have degree ≤ 2, then the graph is A_n.
5. Now, there is a vertex with degree 3 and 3 branches. Since the graph doesn't contain $\widetilde{E}_6$, at least one of the branches length must be 1.
6. If two of the branches have length 1, the graph is D_n.
7. Since the graph doesn't contain $\widetilde{E}_7$, at least one of the other branches' length has to be 2.
8. If the branches' length are $1, 2, 2$ or $1, 2, 3$ or $1, 2, 4$, we get E_6, E_7, E_8, respectively.
9. Since the graph doesn't contain $\widetilde{E}_8$, the branches' length cannot be $1, 2, n$ with $k \geq 5$.

It's a straightforward (albeit tedious) task to check that for $G = A_n, D_n, E_6, E_7, E_8$, the matrix $2I - A_G$ is positive definite (easy to check using Sylvester criterion). Hence, $\lambda_{\max} < 2$ for these graphs. ■

§4 One part of Gabriel's theorem

In this section will will prove one direction of Gabriel's theorem. Let $Q = (Q_0, Q_1, h, t)$ be a quiver, and let $\overline{Q}$ be the corresponding undirected graph. We define an "inner product" on $\mathbb{R}^n$ (where $n = |Q_0|$) as follows:

$$B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \left(2I - A_{\overline{Q}}\right) \mathbf{y}. \tag{8}$$

Note that

$$\begin{aligned}
& \overline{Q} \text{ is a Dynkin diagram} \\
\iff & \text{eigenvalues of } 2I - A_{\overline{Q}} \text{ are positive} \\
\iff & B(\mathbf{x}, \mathbf{x}) > 0 \text{ for every } \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\} \\
\iff & B(\mathbf{x}, \mathbf{x}) > 0 \text{ for every } \mathbf{x} \in \mathbb{Q}^n \setminus \{\mathbf{0}\}, \text{ since } \mathbb{Q} \text{ is dense in } \mathbb{R} \\
\iff & B(\mathbf{x}, \mathbf{x}) > 0 \text{ for every } \mathbf{x} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}, \text{ since we can just multiply it by a positive integer} \\
\iff & B(\mathbf{x}, \mathbf{x}) > 0 \text{ for every } \mathbf{x} \in \mathbb{N}^n \setminus \{\mathbf{0}\}, \text{ since replacing } -m \text{ by } +m \text{ decreases } B(\mathbf{x}, \mathbf{x}).
\end{aligned}$$

Also, notice that

$$B(\mathbf{x}, \mathbf{x}) = 2\sum_{i=1}^{n} x_i^2 - 2 \sum_{\substack{i \text{ and } j \\ \text{are neighbors}}} x_i x_j. \tag{9}$$

So we can rewrite $\frac{1}{2}B(\mathbf{x}, \mathbf{x})$ using the quiver notation as follows:

$$\frac{1}{2}B(\mathbf{x}, \mathbf{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{h(\alpha)} x_{t(\alpha)}. \tag{10}$$

Proposition 4.1

If $Q = (Q_0, Q_1, h, t)$ is a quiver of finite representation type, then $\frac{1}{2}B(\mathbf{x}, \mathbf{x}) > 0$ for every $\mathbf{x} \in \mathbb{N}^n \setminus \{\mathbf{0}\}$.

Proof. Take $\mathbf{m} \in \mathbb{N}^n \setminus \{\mathbf{0}\}$. Since Q is of finite representation type, there are only finitely many representations with dimension vector $\mathbf{m}$.

Consider the following representation: $V_i = k^{m_i}$ for each $i \in Q_0 = \{1, 2, \ldots, n\}$; and if α is an arrow from i to j, then $V_\alpha : k^{m_i} \to k^{m_j}$ is any linear map. Two such representations are isomorphic if and only if there exists $g_i \in \mathrm{GL}(k, m_i)$ for each i such that the following diagram commutes for each arrow α from i to j:

$$\begin{tikzcd}
k^{m_i} \arrow[r, "V_\alpha"] \arrow[d, "g_i"'] & k^{m_j} \arrow[d, "g_j"] \\
k^{m_i} \arrow[r, "W_\alpha"'] & k^{m_j}
\end{tikzcd}$$

In other words,

$$V_\alpha = g_j^{-1} W_\alpha g_i. \tag{11}$$

Let A be the set of all such representations $(V_\alpha)_{\alpha \in Q_1}$, i.e.

$$A = \prod_{\alpha \in Q_1} \mathrm{Hom}\left(k^{m_{t(\alpha)}}, k^{m_{h(\alpha)}}\right); \tag{12}$$

and G be the group

$$G = \prod_{i=1}^{n} \mathrm{GL}(k, m_i).$$

Then G acts on A as (11). Two representations V and W are isomorphic if and only if they are in the same orbit. Since there are only finitely many indecomposable representations, there are only finitely many orbits. Note that, the subgroup

$$G_0 = \left\{ (\lambda \mathbb{1}_{k^{m_i}})_{i \in Q_0} \mid \lambda \in k^* \right\} \tag{13}$$

stablizes every element of A; and it's a 1-dimensional subgroup. Since A is partitioned into finitely many orbits,

$$\dim A = \dim \mathrm{Orb}(a) = \dim G - \dim \mathrm{Stab}(a), \tag{14}$$

for any $a \in A$. $\dim \mathrm{Stab}(a) \geq 1$ as $\mathrm{Stab}(a)$ contains G_0. Hence,

$$\dim A \leq \dim G - 1. \tag{15}$$

Now, $\dim G = \sum_{i \in Q_0} m_i^2$, and $\dim A = \sum_{\alpha \in Q_1} m_{t(\alpha)} m_{h(\alpha)}$. Therefore,

$$\frac{1}{2}B(\mathbf{m}, \mathbf{m}) = \sum_{i \in Q_0} m_i^2 - \sum_{\alpha \in Q_1} m_{h(\alpha)} m_{t(\alpha)} = \dim G - \dim A \geq 1. \tag{16}$$

■

Corollary 4.2

If $Q = (Q_0, Q_1, h, t)$ is a quiver of finite representation type, then its underlying undirected graph $\overline{Q}$ is a Dynkin diagram.

§5 Roots

From this section onwards, we shall try to prove the other direction of Gabriel's theorem. That is, we shall assume that Q is a quiver whose underlying graph $\overline{Q}$ is one of the Dynkin diagrams $(A_n, D_n, E_6, E_7, E_8)$. Then Q has only finitely many indecomposable representations. Recall that, we defined the positive-definite inner product on $\mathbb{R}^n$ as

$$B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \left(2I - A_{\overline{Q}}\right) \mathbf{y}. \tag{17}$$

We've also shown that this is, indeed, a positive definite inner product for Dynkin diagrams.

> **Definition 5.1** (Reflection)**.** Let V be a $\mathbb{R}$-vector space equipped with an inner product $(-,-)$, and $\alpha \in V$. Then the reflection in the hyperplane orthogonal to α is the linear map $s_\alpha : V \to V$, defined as
>
> $$s_\alpha(\beta) = \beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha. \tag{18}$$

> **Definition 5.2** (Root system)**.** Let V be a $\mathbb{R}$-vector space equipped with an inner product $(-,-)$. A finite set $\Phi \subseteq V \setminus \{\mathbf{0}\}$ is said to be a **root system** if the following hold:
>
> 1. $\operatorname{span}\Phi = V$;
> 2. for any $\alpha \in \Phi$, $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$;
> 3. Φ is invariant under the action of any s_α, for any $\alpha \in \Phi$, i.e. $s_\alpha\Phi = \Phi$.
> 4. for any $\alpha, \beta \in \Phi$,
>
> $$n_{\beta,\alpha} := 2\frac{(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}. \tag{19}$$
>
> The elements of Φ are called **roots**, and the dimension of V is called the **rank** of Φ.

Now, consider the set

$$R = \{\mathbf{x} \in \mathbb{Z}^n \mid B(\mathbf{x}, \mathbf{x}) = 2\}. \tag{20}$$

This is a finite set since it's the intersection of a compact space (the sphere of radius $\sqrt{2}$, with respect to the inner product B) and a discrete space $\mathbb{Z}^n$. We claim that this is a root system.

1. Notice that

   $$\boldsymbol{\alpha}_i = (0, \dots, 0, 1, 0, \dots, 0), \tag{21}$$

   with 1 in i-th slot and 0 otherwise, is in R. So $\operatorname{span} R = \mathbb{R}^n$.

2. Given $\mathbf{x} \in R$ and $c \in \mathbb{R}$,

   $$B(c\mathbf{x}, c\mathbf{x}) = c^2 B(\mathbf{x}, \mathbf{x}) = 2c^2. \tag{22}$$

   So $c\mathbf{x} \in R$ if and only if $c = \pm 1$.

3. Given $\mathbf{x}, \mathbf{y} \in R$,

   $$n_{\mathbf{y},\mathbf{x}} := 2\frac{B(\mathbf{y}, \mathbf{x})}{B(\mathbf{x}, \mathbf{x})} = B(\mathbf{y}, \mathbf{x}), \tag{23}$$

   which is clearly an integer since the entries of $\mathbf{x}, \mathbf{y}$ and the matrix $2I - A_{\overline{Q}}$ are all integers.

4. Reflection preserves length. So $B(s_\mathbf{x}\mathbf{y}, s_\mathbf{x}\mathbf{y}) = 2$ for $\mathbf{x}, \mathbf{y} \in R$. Furthermore, the entries of

   $$s_\mathbf{x}\mathbf{y} = \mathbf{y} - n_{\mathbf{y},\mathbf{x}}\mathbf{x} \tag{24}$$

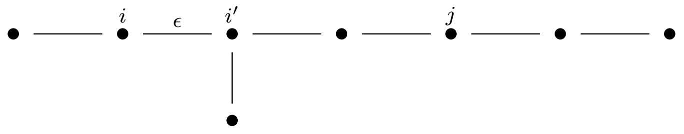
   are all integers. Hence, $s_\mathbf{x} R \subseteq R$. Applying $s_\mathbf{x}$ again, we get $s_\mathbf{x} s_\mathbf{x} R \subseteq s_\mathbf{x} R$. But $s_\mathbf{x}^2 = \mathbb{1}$. So $s_\mathbf{x} R = R$.

So R is a root system. The elements of R will be called roots, and the elements $\boldsymbol{\alpha}_i$ will be called simple roots. Furthermore, notice that the value of $B(\mathbf{x},\mathbf{x})$, for $\mathbf{x} \in \mathbb{Z}^n$ is always an even positive integer. So $B(\mathbf{x},\mathbf{x}) \geq 2$.

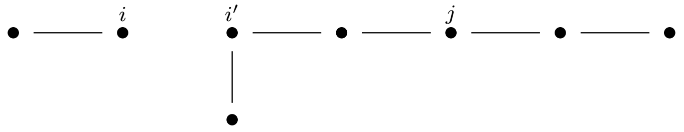
Lemma 5.1

Let $\boldsymbol{\alpha}$ be a root, $\boldsymbol{\alpha} = \sum_{i=1}^{n} k_i \boldsymbol{\alpha}_i$. Then either $k_i \geq 0$ for all i or $k_i \leq 0$ for all i.

Proof. Assume the contrary, i.e., $k_i > 0$, $k_j < 0$. Without loss of generality, we can also assume that $k_s = 0$ for all s between i and j. We can identify the indices i, j with vertices of the graph Γ.



Next, let ϵ be the edge connecting i with the next vertex towards j and i' be the vertex on the other end of ϵ. We then let Γ_1, Γ_2 be the graphs obtained from Γ by removing ϵ. Since Γ is supposed to be a Dynkin diagram—and therefore has no cycles or loops—both Γ_1 and Γ_2 will be connected graphs, which are not connected to each other. (In the diagram below, the left one is Γ_1 and the right one is Γ_2.)



Then we have $i \in \Gamma_1$, $j \in \Gamma_2$. We define

$$\boldsymbol{\beta} = \sum_{m \in \Gamma_1} k_m \boldsymbol{\alpha}_m, \quad \boldsymbol{\gamma} = \sum_{m \in \Gamma_2} k_m \boldsymbol{\alpha}_m. \tag{25}$$

With this choice we get $\boldsymbol{\alpha} = \boldsymbol{\beta} + \boldsymbol{\gamma}$. Since $k_i > 0$, $k_j < 0$ we know that $\boldsymbol{\beta} \neq 0$, $\boldsymbol{\gamma} \neq 0$ and therefore $B(\boldsymbol{\beta},\boldsymbol{\beta}) \geq 2$, $B(\boldsymbol{\gamma},\boldsymbol{\gamma}) \geq 2$. Furthermore,

$$B(\boldsymbol{\beta},\boldsymbol{\gamma}) = -k_i k_{i'}, \tag{26}$$

since Γ_1, Γ_2 are only connected at ϵ. But this has to be a nonnegative number, since $k_i > 0$ and $k_{i'} \leq 0$. This yields

$$B(\boldsymbol{\alpha},\boldsymbol{\alpha}) = B(\boldsymbol{\beta}+\boldsymbol{\gamma},\boldsymbol{\beta}+\boldsymbol{\gamma}) = B(\boldsymbol{\beta},\boldsymbol{\beta}) + 2B(\boldsymbol{\beta},\boldsymbol{\gamma}) + B(\boldsymbol{\gamma},\boldsymbol{\gamma}) \geq 2 + 0 + 2 = 4. \tag{27}$$

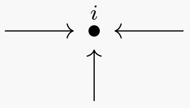
But this is a contradiction, since $\boldsymbol{\alpha}$ was assumed to be a root. ■

Definition 5.3. We call a root $\boldsymbol{\alpha} = \sum_i k_i \boldsymbol{\alpha}_i$ a **positive root** if all $k_i \geq 0$. A root for which $k_i \leq 0$ for all i is called a **negative root**.

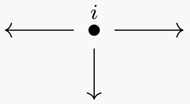
Lemma 5.1 states that every root is either positive or negative.

§6 Reflection Functors

Definition 6.1. Let Q be any quiver. We call a vertex $i \in Q_0$ a **sink** if all edges connected to i point towards i.



We call a vertex $i \in Q_0$ a **source** if all edges connected to i point away from i.



Let Q be any quiver and $i \in Q$ be a sink (a source). Then we let $\overline{Q_i}$ be the quiver obtained from Q by reversing all arrows pointing into (pointing out of) i. We will now define the **reflection functors** (also called **Coxeter functors**).

Definition 6.2. Let Q be a quiver, $i \in Q$ be a sink. Let V be a representation of Q. Then we define the reflection functor

$$F_i^+ : \operatorname{Rep} Q \to \operatorname{Rep} \overline{Q_i}$$

by the rule

$$F_i^+(V)_k = \begin{cases} V_k & \text{if } k \neq i, \\ \operatorname{Ker}\Big(\varphi : \bigoplus_{j\to i} V_j \to V_i\Big) & \text{if } k = i. \end{cases}$$

Also, all maps stay the same but those now pointing out of i. The arrow $i \to k$ (in $\overline{Q_i}$) is represented as the composition

$$\operatorname{Ker}\varphi \hookrightarrow \bigoplus_{j\to i} V_j \twoheadrightarrow V_k.$$

Definition 6.3. Let Q be a quiver, $i \in Q$ be a source. Let V be a representation of Q. Let ψ be the canonical map

$$\psi : V_i \to \bigoplus_{i\to j} V_j.$$

Then we define the reflection functor

$$F_i^- : \operatorname{Rep} Q \to \operatorname{Rep} \overline{Q_i}$$

by the rule

$$F_i^-(V)_k = \begin{cases} V_k & \text{if } k \neq i, \\ \operatorname{Coker}(\psi) = \Big(\bigoplus_{i\to j} V_j\Big) / \operatorname{im}\psi & \text{if } k = i. \end{cases}$$

Again, all maps stay the same but those now pointing into i; The arrow $k \to i$ (in $\overline{Q_i}$) is represented as the composition

$$V_k \hookrightarrow \bigoplus_{i\to j} V_j \twoheadrightarrow \bigoplus_{i\to j} V_j \ / \ \operatorname{im}\psi = \operatorname{Coker}\psi.$$

Proposition 6.1

Let Q be a quiver, V an indecomposable representation of Q.

1. Let $i \in Q$ be a sink. Then either $\dim V_i = 1$, $\dim V_j = 0$ for $j \neq i$; **or** $\varphi : \bigoplus_{j\to i} V_j \to V_i$ is surjective.
2. Let $i \in Q$ be a source. Then either $\dim V_i = 1$, $\dim V_j = 0$ for $j \neq i$; **or** $\psi : V_i \to \bigoplus_{i\to j} V_j$ is injective.

Proof. 1. Choose a complement W of $\operatorname{im}\varphi$, i.e. $V_i = W \oplus \operatorname{im}\varphi$. Then we get

$$\begin{tikzcd} & & 0 \arrow[r] & W & 0 \arrow[l] & & \\ V & = & & & & \oplus & V' \\ & & & 0 \arrow[uu] & & & \end{tikzcd}$$

Since V is indecomposable, one of these summands has to be zero. If the first summand is zero, then φ has to be surjective. If the second summand is zero, then the first one has to be of the desired form $\dim V_i = 1$, $\dim V_j = 0$ for $j \neq i$, because else we could write it as a direct sum of several objects of the type

$$\begin{tikzcd} 0 \arrow[r] & k & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd}$$

which is impossible, since V was supposed to be indecomposable.

2. Follows similarly by splitting away the kernel of ψ.

■

Proposition 6.2

Let Q be a quiver, V be a representation of Q.

1. If
$$\varphi : \bigoplus_{j\to i} V_j \to V_i$$
is surjective, then
$$F_i^- F_i^+ V = V.$$
2. If
$$\psi : V_i \to \bigoplus_{i\to j} V_j$$
is injective, then
$$F_i^+ F_i^- V = V.$$

Proof. In the following proof, we will always mean by $i \to j$ that i points into j in the original quiver Q. We only establish the first statement and we also restrict ourselves to showing that the spaces of V and $F_i^- F_i^+ V$ are the same. It is enough to do so for the i-th space. Let

$$\varphi : \bigoplus_{j\to i} V_j \to V_i$$

be surjective and let

$$K = \ker\varphi.$$

When applying F_i^+, the space V_i gets replaced by K. Furthermore, let

$$\psi : K \to \bigoplus_{j\to i} V_j.$$

After applying F_i^-, K gets replaced by

$$K' = \left(\bigoplus_{j\to i} V_j\right) / (\operatorname{im}\psi).$$

But

$$\operatorname{im}\psi = K$$

and therefore

$$K' = \left(\bigoplus_{j\to i} V_j\right) \Big/ \left(\ker(\varphi : \bigoplus_{j\to i} V_j \to V_i)\right) = \operatorname{im}(\varphi : \bigoplus_{j\to i} V_j \to V_i)$$

by the homomorphism theorem. Since φ was assumed to be surjective, we get

$$K' = V_i.$$

■

Proposition 6.3

Let Q be a quiver, and V be an indecomposable representation of Q. Then F_i^+V and F_i^-V (whenever defined) are either indecomposable or 0.

Proof. We prove the proposition for F_i^+V - the case F_i^-V follows similarly. By Proposition 6.1 it follows that either

$$\varphi : \bigoplus_{j\to i} V_j \to V_i$$

is surjective or $\dim V_i = 1, \dim V_j = 0, \quad j \neq i$. In the last case

$$F_i^+V = 0.$$

So we can assume that φ is surjective. In this case, assume that F_i^+V is decomposable as

$$F_i^+V = X \oplus Y$$

with $X, Y \neq 0$. But F_i^+V is injective at i, since the maps are canonical projections, whose direct sum is the tautological embedding. Therefore X and Y also have to be injective at i and hence (by 6.2)

$$F_i^+F_i^-X = X, \quad F_i^+F_i^-Y = Y$$

In particular

$$F_i^-X \neq 0, \quad F_i^-Y \neq 0.$$

Therefore

$$V = F_i^-F_i^+V = F_i^-X \oplus F_i^-Y$$

which is a contradiction, since V was assumed to be indecomposable. So we can infer that

$$F_i^+V$$

is indecomposable. ■

Proposition 6.4

Let Q be a quiver and V a representation of Q.

1. Let $i \in Q$ be a sink and let V be surjective at i. Then

$$d(F_i^+V) = s_i(d(V)).$$

2. Let $i \in Q$ be a source and let V be injective at i. Then

$$d(F_i^-V) = s_i(d(V)).$$

Proof. We only prove the first statement, the second one follows similarly. Let $i \in Q$ be a sink and let

$$\varphi : \bigoplus_{j \to i} V_j \to V_i$$

be surjective. Let $K = \ker \varphi$. Then

$$\dim K = \sum_{j \to i} \dim V_j - \dim V_i.$$

Therefore we get

$$\left(d(F_i^+ V) - d(V)\right)_i = \sum_{j \to i} \dim V_j - 2 \dim V_i = -B\left(d(V), \alpha_i\right)$$

and

$$\left(d(F_i^+ V) - d(V)\right)_j = 0, \quad j \neq i.$$

This implies

$$\begin{aligned} d(F_i^+ V) - d(V) &= -B\left(d(V), \alpha_i\right) \alpha_i \\ \Leftrightarrow \quad d(F_i^+ V) \;&=\; d(V) - B\left(d(V), \alpha_i\right) \alpha_i \;=\; s_i\left(d(V)\right). \end{aligned}$$

■

§7 Coxeter elements

Definition 7.1. Let Q be a quiver and let Γ be the underlying graph. Fix any labeling $1, \ldots, n$ of the vertices of Γ. Then the Coxeter element c of Q corresponding to this labeling is defined as

$$c = s_1 s_2 \ldots s_n.$$

Lemma 7.1

Let

$$\beta = \sum_i k_i \alpha_i$$

with $k_i \geq 0$ for all i but not all $k_i = 0$. Then there is $N \in \mathbb{N}$, such that

$$c^N \beta$$

has at least one strictly negative coefficient.

Proof. c belongs to a finite group W. So there is $M \in \mathbb{N}$, such that

$$c^M = 1.$$

We claim that

$$1 + c + c^2 + \cdots + c^{M-1} = 0$$

as operators on $\mathbb{R}^n$. This implies what we need, since β has at least one strictly positive coefficient, so one of the elements

$$c\beta, c^2\beta, \ldots, c^{M-1}\beta$$

must have at least one strictly negative one. Furthermore, it is enough to show that 1 is not an eigenvalue for c, since

$$\begin{aligned} (1 + c + c^2 + \cdots + c^{M-1})v = w \neq 0 \\ \Rightarrow \quad cw = c\left(1 + c + c^2 + \cdots + c^{M-1}\right)v = (c + c^2 + c^3 + \cdots + c^{M-1} + 1)v = w. \end{aligned}$$

Assume the contrary, i.e., 1 is a eigenvalue of c and let v be a corresponding eigenvector.

$$cv = v \quad \Rightarrow \quad s_1 \dots s_n v = v$$

$$\Leftrightarrow \quad s_2 \dots s_n v = s_1 v.$$

But since s_i only changes the i-th coordinate of v, we get

$$s_1 v = v \quad \text{and} \quad s_2 \dots s_n v = v.$$

Repeating the same procedure, we get

$$s_i v = v$$

for all i. But this means

$$B(v, \alpha_i) = 0.$$

for all i, and since B is nondegenerate, we get $v = 0$. But this is a contradiction, since v is an eigenvector. ■

§8 Proof of Gabriel's theorem

Let V be an indecomposable representation of Q. We introduce a fixed labeling $1, \dots n$ on Q, such that $i < j$ if one can reach j from i. This is possible, since we can assign the highest label to any sink, remove this sink from the quiver, assign the next highest label to a sink of the remaining quiver and so on. This way we create a labeling of the desired kind.

We now consider the sequence

$$V^{(0)} = V,\ V^{(1)} = F_n^+ V,\ V^{(2)} = F_{n-1}^+ F_n^+ V, \dots$$

This sequence is well defined because of the selected labeling: n has to be a sink of Q, $n-1$ has to be a sink of $\overline{Q_n}$ (where $\overline{Q}_n$ is obtained from Q by reversing all the arrows at the vertex r) and so on. Furthermore, we note that $V^{(n)}$ is a representation of Q again, since every arrow has been reversed twice (since we applied a reflection functor to every vertex). This implies that we can define

$$V^{(n+1)} = F_n^+ V^{(n)}, \dots$$

and continue the sequence to infinity.

Theorem 8.1

There is $m \in \mathbb{N}$, such that

$$d\left(V^{(m)}\right) = \alpha_p$$

for some p.

Proof. If $V^{(i)}$ is surjective at the appropriate vertex k, then

$$d\left(V^{(i+1)}\right) = d\left(F_k^+ V^{(i)}\right) = s_k d\left(V^{(i)}\right).$$

This implies, that if $V^{(0)}, \dots, V^{(i-1)}$ are surjective at the appropriate vertices, then

$$d\left(V^{(i)}\right) = \dots s_{n-1} s_n d(V).$$

By Lemma 7.1 this cannot continue indefinitely - since $d\left(V^{(i)}\right)$ may not have any negative entries. Let i be smallest number such that $V^{(i)}$ is not surjective at the appropriate vertex. By Proposition 6.3 it is indecomposable. So, by Proposition 6.1, we get

$$d(V^{(i)}) = \alpha_p$$

for some p. ■

We are now able to prove Gabriel's theorem. Namely, we get the following corollaries.

Corollary 8.2

Let Q be a quiver, V be any indecomposable representation. Then $d(V)$ is a positive root.

Proof. By Theorem 8.1

$$s_{i_1} \dots s_{i_m} (d(V)) = \alpha_p.$$

Since the s_i preserve B, we get

$$B(d(V), d(V)) = B(\alpha_p, \alpha_p) = 2.$$

■

Corollary 8.3

Let V, V' be indecomposable representations of Q such that $d(V) = d(V')$. Then V and V' are isomorphic.

Proof. Let i be such that

$$d\left(V^{(i)}\right) = \alpha_p.$$

Then we also get $d\left(V'^{\ (i)}\right) = /\alpha_p$. So

$$V'^{\ (i)} = V^{(i)} =: V^i.$$

Furthermore we have

$$V^{(i)} = F_k^+ \dots F_{n-1}^+ F_n^+ V^{(0)}$$
$$V'^{\ (i)} = F_k^+ \dots F_{n-1}^+ F_n^+ V'^{(0)}.$$

But both $V^{(i-1)}, \dots, V^{(0)}$ and $V'^{\ (i-1)}, \dots, V'^{\ (0)}$ have to be surjective at the appropriate vertices. This implies

$$F_n^- F_{n-1}^- \dots F_k^- V^i = \begin{cases} F_n^- F_{n-1}^- \dots F_k^- F_k^+ \dots F_{n-1}^+ F_n^+ V^{(0)} & = V^{(0)} & = V \\ F_n^- F_{n-1}^- \dots F_k^- F_k^+ \dots F_{n-1}^+ F_n^+ V'^{\ (0)} & = V'^{\ (0)} & = V' \end{cases}$$

■

These two corollaries show that there are only finitely many indecomposable representations (since there are only finitely many roots) and that the dimension vector of each of them is a positive root. The last statement of Gabriel's theorem follows from

Corollary 8.4

For every positive root α, there is an indecomposable representation V with

$$d(V) = \alpha.$$

Proof. Consider the sequence

$$s_n\alpha, s_{n-1}s_n\alpha, \dots$$

Consider the first element of this sequence which is a negative root (this has to happen by Lemma 7.1) and look at one step before that, calling this element β. So β is a positive root and $s_i\beta$ is a negative root for some i. But since the s_i only change one coordinate, we get

$$\beta = \alpha_i$$

and

$$(s_q \dots s_{n-1} s_n)\alpha = \alpha_i.$$

We let $k_{(i)}$ be the representation having dimension vector α_i. Then we define

$$V = F_n^- F_{n-1}^- \dots F_q^- k_{(i)}.$$

This is an indecomposable representation and

$$d(V) = \alpha.$$

■

§9 Kac's theorem

The heart of the proof of Gabriel's theorem was that if α is a positive root, then there is exactly one indecomposable representation of dimension vector α. There are finitely many such positive roots since the root system consists of elements of $\mathbb{Z}^n$ having norm $\sqrt{2}$, with respect to the inner product B.

Now, when Q is not of ADE-type, then B is not a positive-definite inner product. So we cannot say that there are finitely many roots in that case. However, it still holds that there is exactly one indecomposable representation corresponding to each positive (real) root. The precise statement of the result is given by Kac's theorem.

Theorem 9.1 (Kac's Theorem)

For a quiver Q,

1. There exists an indecomposable representation of dimension α if and only if $\alpha \in \Phi^+$, the set of positive roots of Q.
2. If $\alpha \in \Phi^+_{\text{re}}$, the set of positive real roots, then there exists exactly one indecomposable representation of dimension α.
3. If $\alpha \in \Phi^+_{\text{im}}$, the set of positive imaginary roots, then there exist infinitely many indecomposable representations of dimension α.

Here $\Phi^+, \Phi^+_{\text{re}}, \Phi^+_{\text{im}}$ denote the sets of positive roots, positive real roots, and positive imaginary roots, respectively (to be defined shortly). In this paper, we will prove a weaker version of Kac's Theorem, which we will refer to as the *weak Kac's theorem*. It will mainly use deformed preprojective algebras and properties of roots.

Theorem 9.2 (Weak Kac's Theorem)

Suppose α is indivisible. Then there exists an indecomposable representation of dimension α if and only if $\alpha \in \Phi^+$.

§10 Roots and preprojective algebras

For each vertex $x \in Q_0$, let ε_x denote the standard basis vector in $\mathbb{R}^{Q_0}$ (or $\mathbb{C}^{Q_0}$) corresponding to x. We define the reflection $\sigma_x \colon \mathbb{R}^{Q_0} \to \mathbb{R}^{Q_0}$ by

$$\sigma_x(\alpha) = \alpha - (\alpha, \varepsilon_x)\varepsilon_x, \quad \text{for } \alpha \in \mathbb{R}^{Q_0},$$

where $(\cdot,\cdot)$ denotes the symmetric bilinear form associated with the quiver, which is the same as $B(\cdot,\cdot)$ introduced earlier. This reflection can be extended linearly to $\alpha \in \mathbb{C}^{Q_0}$ as

$$(\sigma_x^* \lambda)(y) = \lambda(y) - (\varepsilon_x, \varepsilon_y)\lambda(x),$$

for all $\lambda \in \mathbb{C}^{Q_0}$ and $y \in Q_0$.

We can now define the set of real roots as

$$\Phi_{\text{re}} = \bigcup_{x \in Q_0} W\varepsilon_x,$$

where W is the Weyl group generated by the reflections σ_x. The positive and negative real roots are given respectively by

$$\Phi_{\text{re}}^{+} = \Phi_{\text{re}} \cap \mathbb{N}^{Q_0}, \qquad \Phi_{\text{re}}^{-} = \Phi_{\text{re}} \cap (-\mathbb{N}^{Q_0}).$$

To define the set of imaginary roots, we first introduce the notion of support. For $\alpha \in \mathbb{N}^{Q_0}$, define the *support* of α, denoted $\operatorname{supp}(\alpha)$, as the set of all nonzero dimension vectors β such that $\alpha - \beta$ is also a nonzero dimension vector.

Let $K \subseteq \mathbb{N}^{Q_0}$ be the set of all nonzero dimension vectors α such that the support of α is connected and

$$(\alpha, \varepsilon_x) \leq 0 \quad \text{for all } x \in Q_0.$$

Definition 10.1 (Imaginary Roots). Define

$$\Phi_{\text{im}}^{+} = WK, \qquad \Phi_{\text{im}}^{-} = -\Phi_{\text{im}}^{+}, \qquad \Phi_{\text{im}} = \Phi_{\text{im}}^{+} \cup \Phi_{\text{im}}^{-}.$$

Note that $(\alpha, \alpha) = (\varepsilon_x, \varepsilon_x) = 2$ for all $\alpha \in \Phi_{\text{re}}$, while $(\alpha, \alpha) \leq 0$ for all $\alpha \in \Phi_{\text{im}}$. Hence, $\Phi_{\text{re}} \cap \Phi_{\text{im}} = \varnothing$.

We will prove a weak version of Kac's Theorem for the particular case where α is *indivisible*, meaning that it cannot be written as $\alpha = k\alpha_0$ for any $k \in \mathbb{N} \setminus \{1\}$ and some dimension vector α_0. We then show that there exists an indecomposable representation of dimension α if and only if

$$\alpha \in \Phi^{+} = \Phi_{\text{im}}^{+} \cup \Phi_{\text{re}}^{+}.$$

The first step in proving the weak Kac's Theorem will be to show that the existence of an indecomposable representation of indivisible dimension α is independent of orientation. For Gabriel's Theorem, we established this when Q is acyclic, without any restriction on the dimension vector.

Let $Q = (Q_0, Q_1, h, t)$ be a quiver, where $h, t \colon Q_1 \to Q_0$ are the head and tail maps, respectively. We define the *double quiver* $\overline{Q} = (Q_0, \overline{Q}_1, h, t)$ as follows: for each arrow $a \in Q_1$, define an opposite arrow a^* such that

$$h(a^*) = t(a), \qquad t(a^*) = h(a).$$

Then we set

$$Q_1^* = \{a^* \mid a \in Q_1\}, \qquad \overline{Q}_1 = Q_1 \cup Q_1^*.$$

The *opposite quiver* Q^* is the quiver with the same set of vertices Q_0 and arrows Q_1^* only.

Definition 10.2 (Deformed Preprojective Algebra). For $\lambda \in \mathbb{C}^{Q_0}$, define the *deformed preprojective algebra* by

$$\Pi_\lambda = \mathbb{C}\overline{Q}/(r_\lambda),$$

where

$$r_\lambda = \sum_{a \in Q_1} (a^*a - aa^*) - \lambda,$$

viewed as an element of $\mathbb{C}\overline{Q}$ by identifying λ with $\sum_{x \in Q_0} \lambda(x) e_x$, and e_x are the primitive idempotents corresponding to vertices $x \in Q_0$.

A useful way to interpret $\operatorname{Rep}_\alpha(\Pi_\lambda)$ is as a subset of

$$\operatorname{Rep}_\alpha(Q) \oplus \operatorname{Rep}_\alpha(Q^*),$$

consisting of pairs of representations (V, W) satisfying the relation

$$\sum_{\substack{a\in Q_1\\ t(a)=x}} W(a^*)V(a) - \sum_{\substack{a\in Q_1\\ h(a)=x}} V(a)W(a^*) = \lambda(x)\,\mathbf{1}_{\alpha(x)}, \quad \text{for all } x \in Q_0.$$

This perspective will make the following definitions and constructions particularly useful.

Definition 10.3 (The Moment Map)**.** The *moment map*

$$\mu_\alpha : \mathrm{Rep}_\alpha(Q) \oplus \mathrm{Rep}_\alpha(Q^*) \longrightarrow \bigoplus_{x\in Q_0} \mathrm{End}(\mathbb{C}^{\alpha(x)})$$

is defined by

$$\mu_\alpha(V, W) = \sum_{\substack{a\in Q_1\\ t(a)=x}} W(a^*)V(a) - \sum_{\substack{a\in Q_1\\ h(a)=x}} V(a)W(a^*).$$

This makes $\mu_\alpha^{-1}(\lambda) = \mathrm{Rep}_\alpha(\Pi_\lambda)$, where λ has been identified with $(\lambda(x)\mathbf{1}_{\alpha(x)}, x \in Q_0)$. Note that we continue to use this identification.

Lemma 10.1

If $\lambda(\alpha) \neq 0$, then $\mathrm{Rep}_\alpha(\Pi_\lambda) = \varnothing$.

Proof. We note that if $\mu_\alpha(V, W) = (A(x), x \in Q_0)$, then

$$\sum_{x\in Q_0} \mathrm{Tr}(A(x)) = \sum_{a\in Q_1} \mathrm{Tr}(W(a^*)V(a) - V(a)W(a^*)) = 0.$$

Hence, if $(V, W) \in \mu_\alpha^{-1}(\lambda)$, it follows that

$$0 = \sum_{x\in Q_0} \lambda(x)\alpha(x) = \lambda(\alpha).$$

■

Let V be a representation of Q of dimension α. Then the following sequence is exact:

$$0 \longrightarrow \mathrm{Hom}_Q(V, V) \longrightarrow \bigoplus_{x\in Q_0} \mathrm{End}(\mathbb{C}^{\alpha(x)}) \longrightarrow \bigoplus_{a\in Q_1} \mathrm{Hom}(V(t(a)), V(h(a))) \longrightarrow \mathrm{Ext}^1_Q(V, V) \longrightarrow 0.$$

Note that $\mathrm{End}(\mathbb{C}^{\alpha(x)}) \cong \mathrm{Hom}_{\mathbb{C}}(V(x), V(x))$. By dualizing and identifying

$$\mathrm{Rep}_\alpha(Q^*) = \bigoplus_{a\in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{h(a)}, \mathbb{C}^{t(a)}) \cong \bigoplus_{a\in Q_1} \mathrm{Hom}_{\mathbb{C}}(V(t(a)), V(h(a)))^*,$$

and identifying $\mathrm{End}(\mathbb{C}^{\alpha(x)})$ with $\mathrm{End}(\mathbb{C}^{\alpha(x)})^*$, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_Q(V, V)^* \longrightarrow \mathrm{Rep}_\alpha(Q^*) \xrightarrow{\mu_\alpha^V} \bigoplus_{x\in Q_0} \mathrm{End}(\mathbb{C}^{\alpha(x)}) \xrightarrow{\gamma} \mathrm{Hom}_Q(V, V)^* \longrightarrow 0,$$

where the map $\mu_\alpha^V = \mu_\alpha(V, -) : \mathrm{Rep}_\alpha(Q^*) \to \bigoplus_{x\in Q_0} \mathrm{End}(\mathbb{C}^{\alpha(x)})$.

The last result we need to prove for the independence of orientation is the following. By an *indecomposable summand* of V, we mean an indecomposable representation W such that $V = W \oplus Y$ for some representation Y.

Theorem 10.2

We have

$$(\mu_\alpha^V)^{-1}(\lambda) \neq \varnothing \quad \Longleftrightarrow \quad \lambda(\dim Y) = 0 \text{ for every indecomposable summand } Y \text{ of } V.$$

Proof. If $(\mu_\alpha^V)^{-1}(\lambda) \neq \varnothing$, then by exactness we have $\gamma(\lambda) = \lambda(\alpha) = 0$. Let $p : V \to V$ be the projection onto an indecomposable summand Y. Then

$$0 = \gamma(\lambda)(p) = \sum_{x \in Q_0} \lambda(x) \operatorname{Tr}(p(x)) = \lambda(\dim Y).$$

Conversely, suppose V is indecomposable and $\lambda(\alpha) = 0$. We will show that $\gamma(\lambda) = 0$, which by exactness implies $(\mu_\alpha^V)^{-1}(\lambda) \neq \varnothing$. The endomorphisms of V are spanned by the identity and nilpotent endomorphisms, so it suffices to check these two cases. We already know $\gamma(\lambda)(1_V) = \lambda(\alpha) = 0$. If $f \in \operatorname{End}_Q(V)$ is nilpotent, then

$$\gamma(\lambda)(f) = \sum_{x \in Q_0} \operatorname{Tr}(\lambda(x) f(x)) = 0,$$

since the trace of a nilpotent matrix is 0. Thus $\gamma(\lambda) = 0$ and therefore $\mu_\alpha^{-1}(\lambda) \neq \varnothing$.

If $V = V_1 \oplus \cdots \oplus V_r$ with each V_i indecomposable of dimension α_i, then by the previous case there exist $(V_i, W_i) \in \operatorname{Rep}_{\alpha_i}(\Pi_\lambda)$. Hence $(V, W_1 \oplus \cdots \oplus W_r) \in \mu_\alpha^{-1}(\lambda)$. ■

We can now conclude that orientation does not affect the existence of an indecomposable representation. This theorem is also the main reason why we will need to restrict to indivisible dimension vectors.

Theorem 10.3

Suppose that α is an indivisible dimension vector. Whether there exists an indecomposable representation of Q of dimension α is independent of the orientation of Q.

Proof. Let Q and Q' be quivers that differ only by orientation. Then their double quivers $\overline{Q}$ and $\overline{Q}'$ are the same. We first show that we can choose $\lambda \in \mathbb{C}^{Q_0}$ such that for all $\beta \in \mathbb{Z}^{Q_0}$, we have $\lambda(\beta) = 0$ if and only if β is a rational multiple of α.

Let $n = |Q_0|$. Choose $\delta^{(1)}, \dots, \delta^{(n-1)} \in \mathbb{Q}^n$ such that

$$\{c \in \mathbb{Q}^n \mid \delta^{(1)}(c) = \cdots = \delta^{(n-1)}(c) = 0\}$$

is the $\mathbb{Q}$-span of α. Let $\lambda = t_1\delta^{(1)} + \cdots + t_{n-1}\delta^{(n-1)}$ where $t_1, \dots, t_{n-1} \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$.

Clearly, $\lambda(\beta) = 0$ when $\beta = t\alpha$ for some $t \in \mathbb{Q}$. Now assume $\lambda(\beta) = 0$. Since the t_i are independent, it follows that $\delta^{(i)}(\beta) = 0$ for each i, so β lies in the $\mathbb{Q}$-span of α. Thus $\beta = t\alpha$.

Let V be an indecomposable representation of Q of dimension α. Then V can be lifted to a representation Z of $\overline{Q}$. Restricting Z to Q' gives a representation V'. It follows by Theorem 10.2 that $\lambda(\dim Y) = 0$. By the above remark, this implies that $\dim Y$ is a rational multiple of α. Since α is indivisible, we must have $\dim Y = 0$ or $\dim Y = \alpha$. Thus V' is indecomposable. ■

§11 Proof of weak Kac's theorem

We start by establishing some cases when there must exist an indecomposable representation of a certain dimension.

Lemma 11.1

Let α be an indivisible dimension vector. Suppose that there exists an indecomposable representation V of Q of dimension α. Then either $\alpha = \varepsilon_x$ and $V \cong S_x$, or $\sigma_x(\alpha) \in \mathbb{N}^{Q_0}$

and there exists an indecomposable representation of dimension $\sigma_x(\alpha)$.

Proof. Let Q' be a quiver differing from Q only by orientation, where x is a sink. Then we know that there exists a representation V' of Q' of dimension α. The statement then follows directly from Proposition 6.3.$F_x^+(V') = 0$ or not. ■

Theorem 11.2

Suppose $\alpha \in \mathbb{N}^{Q_0}$ is indivisible, and $\beta = w(\alpha)$ for some $w \in W$. Then there exists an indecomposable representation of dimension α if and only if there exists an indecomposable representation of dimension β or $-\beta$.

Proof. Write w as $w = \sigma_{x_r} \cdots \sigma_{x_1}$. We use induction on r to prove the statement. When $r = 0$, the result is clear.

Assume by induction that there exists an indecomposable representation of dimension γ, where

$$\gamma = \pm\sigma_{x_{r-1}} \cdots \sigma_{x_1}(\alpha), \quad \text{and} \quad \beta = \pm\sigma_{x_r}(\gamma).$$

We apply Lemma 11.1. First, suppose that $\sigma_{x_r}(\gamma) \notin \mathbb{N}^{Q_0}$ and $\gamma = \varepsilon_{x_r}$. Then $\sigma_{x_r}(\gamma) = -\varepsilon_{x_r}$, so the simple representation S_{x_r} is an indecomposable representation of dimension $\pm\beta$.

Now suppose that $\sigma_{x_r}(\gamma) \in \mathbb{N}^{Q_0}$. Then directly from Lemma 11.1, there exists an indecomposable representation of dimension $\sigma_{x_r}(\gamma) = \pm\beta$. ■

We now prove the results we need about the roots in order to establish the weak Kac's Theorem using the previous two lemmas.

Lemma 11.3

For every $\alpha \in \Phi^+_{\text{re}}$, there exists an indecomposable representation of dimension α, and moreover,

$$\Phi_{\text{re}} = \Phi^+_{\text{re}} \cup \Phi^-_{\text{re}}.$$

Proof. By definition, we know that $\alpha = w(\varepsilon_x)$ for some $w \in W$. From Theorem 11.2, we know that there exists an indecomposable representation with dimension α or $-\alpha$. Since α is assumed to be positive, it must correspond to a representation of dimension α. ■

Lemma 11.4

If $\alpha \in K$, then there exist infinitely many indecomposable representations of dimension α.

Proof. In order to prove this lemma, we need another result:

Lemma 11.5

Suppose that α is a dimension vector such that $\alpha(x) > 0$ for all $x \in Q_0$, and $(\alpha, \varepsilon_x) \leq 0$ for all $x \in Q_0$. If $\beta \in \operatorname{supp}(\alpha)$, then we have $(\beta, \alpha - \beta) \leq 0$. Moreover, if $(\beta, \alpha - \beta) = 0$, then β is proportional to α and $(\alpha, \varepsilon_x) = 0$ for all x.

Proof. Without loss of generality we may assume that the support of α is Q_0. We have

$$\begin{aligned} 0 &\geq \frac{1}{2}\sum_{x,y}\left(\frac{\beta(x)}{\alpha(x)} - \frac{\beta(y)}{\alpha(y)}\right)^2 \alpha(x)\alpha(y)(\varepsilon_x, \varepsilon_y) \\ &= \sum_{x,y}\frac{\beta(x)^2\alpha(y)}{\alpha(x)}(\varepsilon_x, \varepsilon_y) - \sum_{x,y}\beta(x)\beta(y)(\varepsilon_x, \varepsilon_y) \qquad (28) \\ &= \sum_{x}\frac{\beta(x)^2}{\alpha(x)}(\varepsilon_x, \alpha) - (\beta, \beta). \end{aligned}$$

Replacing β by $\alpha-\beta$ in (28) and then adding that to (28) again gives

$$\begin{aligned}0 &\geq \sum_x \frac{\beta(x)^2+(\alpha(x)-\beta(x))^2}{\alpha(x)}(\varepsilon_x,\alpha)-(\beta,\beta)-(\alpha-\beta,\alpha-\beta)\\ &=\sum_x \alpha(x)(\varepsilon_x,\alpha)-2\sum_x\frac{\beta(x)(\alpha(x)-\beta(x))}{\alpha(x)}(\varepsilon_x,\alpha)-(\beta,\beta)-(\alpha-\beta,\alpha-\beta)\\ &\geq (\alpha,\alpha)-(\beta,\beta)-(\alpha-\beta,\alpha-\beta)=2(\alpha-\beta,\beta)\end{aligned}\tag{29}$$

because $(\varepsilon_x,\alpha)\leq 0$ for all $x\in Q_0$.

Suppose that $(\beta,\alpha-\beta)=0$. Then we must have equality in (28), so β is proportional to α. We also have equality in (29), so $(\varepsilon_x,\alpha)=0$ for all $x\in Q_0$. $\blacksquare$

Now we get back to the proof of Lemma 11.4. Without loss of generality we may assume that the support of α is Q_0, i.e. $\alpha(x)>0$ for all $x\in Q_0$. So Q is a connected quiver.

Suppose that $\beta\in\operatorname{supp}(\alpha)$. We can view $\operatorname{Rep}_\beta(Q)\oplus\operatorname{Rep}_{\alpha-\beta}(Q)$ as a subspace of $\operatorname{Rep}_\alpha(Q)$. Consider the morphism

$$\psi_\beta:\mathrm{GL}_\alpha\times\operatorname{Rep}_\beta(Q)\oplus\operatorname{Rep}_{\alpha-\beta}(Q)\longrightarrow\operatorname{Rep}_\alpha(Q)$$

defined by $(A,V,W)\mapsto A\cdot(V\oplus W)$. Let Z_β be the Zariski closure of the image of ψ. The tangent space of GL_α at the identity 1_α is naturally isomorphic to $\bigoplus_{x\in Q_0}\operatorname{End}(\mathbb{C}^{\alpha(x)})$ and the tangent spaces of $\operatorname{Rep}_\alpha(Q)$ can be identified with $\operatorname{Rep}_\alpha(Q)$ itself. The tangent map $(d\psi_\beta)_{(1,V,W)}$ at $(1,V,W)$ is given by

$$((A(x),x\in Q_0),V',W')\longmapsto\left(\begin{pmatrix}V'(a)&0\\0&W'(a)\end{pmatrix}+A(ha)\begin{pmatrix}V(a)&0\\0&W(a)\end{pmatrix}-\begin{pmatrix}V(a)&0\\0&W(a)\end{pmatrix}A(ta),\ a\in Q_1\right).$$

If the $(A(x),x\in Q_0)$ has the block form

$$\begin{pmatrix}B&0\\0&C\end{pmatrix},$$

then the image lies in $\operatorname{Rep}_\beta(Q)\oplus\operatorname{Rep}_{\alpha-\beta}(Q)$. So the dimension of the image of the tangent map is at most

$$\begin{aligned}&\dim\operatorname{Rep}_\beta(Q)+\dim\operatorname{Rep}_{\alpha-\beta}(Q)+\sum_{x\in Q_0}(\alpha(x)^2-\beta(x)^2-(\alpha(x)-\beta(x))^2)\\ &=-\langle\beta,\beta\rangle-\langle\alpha-\beta,\alpha-\beta\rangle+\langle\alpha,\alpha\rangle+\dim\operatorname{Rep}_\alpha(Q)=\dim\operatorname{Rep}_\alpha(Q)+(\beta,\alpha-\beta).\end{aligned}$$

So the tangent map induces a linear map

$$\bigoplus_{x\in Q_0}\frac{\operatorname{End}(\mathbb{C}^{\alpha(x)})}{\operatorname{End}(\mathbb{C}^{\beta(x)})\oplus\operatorname{End}(\mathbb{C}^{\alpha(x)-\beta(x)})}\longrightarrow\frac{\operatorname{Rep}_\alpha(Q)}{\operatorname{Rep}_\beta(Q)\oplus\operatorname{Rep}_{\alpha-\beta}(Q)}.$$

The codimension of the image is at least

$$\begin{aligned}&\dim\operatorname{Rep}_\alpha(Q)-\dim\operatorname{Rep}_\beta(Q)-\dim\operatorname{Rep}_{\alpha-\beta}(Q)-\sum_{x\in Q_0}(\alpha(x)^2-\beta(x)^2-(\alpha-\beta)(x)^2)\\ &=-\langle\alpha,\alpha\rangle+\langle\beta,\beta\rangle+\langle\alpha-\beta,\alpha-\beta\rangle=-(\beta,\alpha-\beta).\end{aligned}$$

We have $(\beta,\alpha-\beta)\geq 0$ for all $\beta\in\operatorname{supp}(\alpha)$ by Lemma 11.5.

Suppose that $(\beta,\alpha-\beta)=0$ for some $\beta\in\operatorname{supp}(\alpha)$. Then β is proportional to α and $(\alpha,\varepsilon_x)=0$ for all $x\in Q_0$ by Lemma 11.5. In particular we have $(\alpha,\alpha)=0$. If γ is any dimension vector, then there exists a positive integer m such that $m\alpha-\gamma$ is a dimension vector. So we have

$$-m(\gamma,\gamma)=(\gamma,m\alpha-\gamma)\leq 0.$$

This shows that $(\gamma, \gamma) \geq 0$ for all dimension vectors. So Q must be an extended Dynkin quiver. In this case we already know that there exists an indecomposable representation of dimension α.

Suppose that $(\beta, \alpha - \beta) > 0$ for all $\beta \in \operatorname{supp}(\alpha)$. Then Z_β has codimension ≥ 1 for all $\beta \in \operatorname{supp}(\alpha)$. Any representation in

$$\operatorname{Rep}_\alpha(Q) \setminus \bigcup_{\beta \in \operatorname{supp}(\alpha)} Z_\beta$$

is indecomposable. ■

Theorem 11.6 (weak Kac's Theorem)

Suppose that α is indivisible. Then there exists an indecomposable representation of dimension α if and only if $\alpha \in \Phi^+$.

Proof. We have already shown that if $\alpha \in \Phi^+_{\text{re}}$, then there exists an indecomposable representation of dimension α in Lemma 11.3. We now show that if $\alpha \in \Phi^+_{\text{im}}$ and is indivisible, then there exists an indecomposable representation of dimension α.

Let $\beta \in K$ such that $w(\beta) = \alpha$ for $w = \sigma_{x_r} \cdots \sigma_{x_1} \in \mathcal{W}$. This exists by definition. Note that if β was not indivisible and was equal to $\beta = s\beta'$, then $w(\beta) = w(s\beta') = sw(\beta') = \alpha$. This would contradict α being indivisible.

We will now use induction on r. If $r = 0$, then $\beta = \alpha$ and the statement is true. Now assume the statement holds for $\gamma = \sigma_{x_{r-1}} \cdots \sigma_{x_1}(\beta)$. We apply Lemma 4.8. If $\alpha = \sigma_{x_r}(\gamma) \notin \mathbb{N}^{Q_0}$, then $\gamma = \varepsilon_{x_r}$. Then $\alpha \in \Phi_{\text{re}} \cap \Phi_{\text{im}} = \varnothing$, a contradiction. Hence $\alpha \in \mathbb{N}^{Q_0}$ and there exists an indecomposable representation of dimension α.

The final step is to show that if there exists an indecomposable representation of dimension α, then $\alpha \in \Phi^+$. We show this by induction on $|\alpha| = \sum_{x \in Q_0} \alpha(x)$. If $|\alpha| = 1$, then this must be a simple representation.

Suppose $|\alpha| > 1$ and there exists an indecomposable representation of dimension α. If there exists $x \in Q_0$ such that $(\alpha, \varepsilon_x) > 0$, then

$$|\sigma_x \alpha| = |\alpha| - (\alpha, \varepsilon_x) < |\alpha|.$$

By our induction hypothesis, $\sigma_x \alpha \in \Phi^+$. Then $\sigma_x^2 \alpha = \alpha$ implies $\alpha \in \Phi^+$.

Suppose instead that $(\alpha, \varepsilon_x) \leq 0$ for all $x \in Q_0$. We also know that the support of α is connected, since otherwise we could decompose it. Therefore $\alpha \in K$, so by Lemma 11.4 there exists an indecomposable representation of dimension α. ■

References

[1] Michel Brion. “Representations of Quivers”. In: Lecture notes, Institut Fourier, Grenoble. 2010. URL: `https://www-fourier.ujf-grenoble.fr/~mbrion/notes_quivers_rev.pdf`.

[2] William Crawley-Boevey. *Lectures on Representations of Quivers*. Tech. rep. Lecture notes from a course delivered at Oxford in spring 1992, 38 pages. Mathematical Institute, University of Oxford, 1992. URL: `https://www.math.uni-bielefeld.de/~wcrawley/quivlecs.pdf`.

[3] Harm Derksen and Jerzy Weyman. *An Introduction to Quiver Representations*. Vol. 184. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 2017, p. 344. ISBN: 978-1-4704-2556-2.

[4] Pavel Etingof et al. *Introduction to Representation Theory*. Vol. 59. Student Mathematical Library. Providence, RI: American Mathematical Society, 2011, p. 228. ISBN: 978-0-8218-5351-1.

[5] I. N. Bernstein, I. M. Gelfand, and V. A. Ponomarev. “Coxeter functors and Gabriel’s theorem”. In: *Russian Mathematical Surveys* 28.2 (1973), pp. 17–32. DOI: `10.1070/RM1973v028n02ABEH001526`.

[6] Ralf Schiffler. *Quiver Representations*. 1st ed. CMS Books in Mathematics. Cham, Switzerland: Springer International Publishing, 2014, pp. xi + 230. ISBN: 978-3-319-09203-4. DOI: `10.1007/978-3-319-09204-1`.

Approval

The internship report titled *“Quiver Representation: Gabriel’s Theorem and Kac’s Theorem”* submitted by **Atonu Roy Chowdhury**, a participant of the ICTP PWF: Physics for Bangladesh Online Summer Internship, has been found satisfactory in partial fulfillment of the requirements of the internship program. The internship was conducted under the supervision of **Ahmed Ittihad Hasib** during the period 15 July 2025 to 15 October 2025.

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