Internship Report

q-deformation of $U(\mathfrak{sl}_2)$, their Representations and the Quanum Yang Baxter Equation

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Contents

Notation

Let us fix an invertible element $q \in k$ of some ground field k with $q \neq 0$ and $q^2 \neq 1$. We introduce the following notation:

$$
\begin{aligned}
[n]_q &= \frac{q^n - q^{-n}}{q - q^{-1}}, \\
[-n]_q &= -[n]_q, \\
[m+n]_q &= q^n [m]_q + q^{-m}[n]_q, \\
\begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n]_q!}{[k]_q!\,[n-k]_q!}, \\
(x+y)_q^n &= \sum_{k=0}^{n} q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}.
\end{aligned}
$$

1 Introduction

The theory of quantum groups emerged in the 1980s when physicists were working on integrable systems and the quantum inverse scattering method. The key object was the Quantum Yang-Baxter Equation which arose in studies of statistical mechanics. Physicists discovered that solutions to this equation led to exactly solvable models. In the early 1980s, the Leningrad School, particularly Faddeev, Sklyanin, and Takhtajan[3], developed a framework known as the Quantum Inverse Scattering Method which produced new algebraic structures related to these solutions.

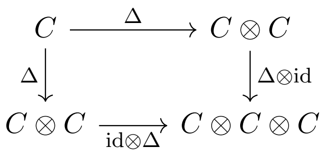
It was Vladimir Drinfeld[2] and Michio Jimbo[7] who, independently around 1985, formalized this concept by defining quantum groups as a new type of Hopf algebras. They showed that these structures are not groups in the traditional sense but rather deformations, which are sometimes called the q-deformation, of classical algebraic structures, for example the universal enveloping algebras of some Lie algebras. This q-deformation is a process that modifies the algebraic relations by introducing a parameter q. When q is set to 1, the original Lie algebra structure is recovered. In this report we will explore the q-deformations of the Universal enveloping algebra of $U(\mathfrak{sl}_2)$ and a systematic way to recover the classical universal enveloping algebra.

2 Hopf Algebras

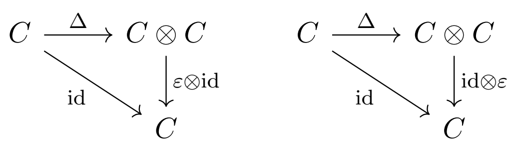
2.1 Coalgebra

We know that an Algebra A is a vector space together with a multiplication $m : A \otimes A \to A$ and a unit element $u : k \to A$ which satisfy the associativity and unitality property. To define coalgebras we will simply reverse the arrrows in every direction that is we define a notion of "comultiplication" $\Delta : A \to A \otimes A$ and "counit" $\varepsilon : A \to k$ satisfying the following axioms:

(1) (Coassociativity) The following diagram commutes



(2) (Counitality) The followings are equal



For an element $c \in C$ we write the coproduct in terms of the tensors $\Delta(c) = \sum_i c^i_{(1)} \otimes c^i_{(2)}$. We will use the Sweedler's notaton where we drop the sum symbol and write

$$\Delta(c) = c_{(1)} \otimes c_{(2)}$$

. Using the Sweedler's notation, the counit axioms are

$$c_{(1)}\varepsilon(c_{(2)}) = \varepsilon(c_{(1)})c_{(2)}$$

The coassociativity condition states that

$$(c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)} = c_{(1)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)}$$

2.2 Bialgebra

A Bialgebra is a vector sapce B with an algebra structure (B, m, u) and a coalgebra structure (B, Δ, ε) satisfying the following axioms

(1) The following diagram commutes

$$\begin{array}{ccccc} B \otimes B & \xrightarrow{m} & B & \xrightarrow{\Delta} & B \otimes B \\ \downarrow{\scriptstyle \Delta\otimes\Delta} & & & & \downarrow{\scriptstyle m\otimes m} \\ B \otimes B \otimes B \otimes B & & \xrightarrow[\mathrm{id}\otimes\sigma\otimes\mathrm{id}]{} & & B \otimes B \otimes B \otimes B \end{array}$$

where for a pait of vector space V, W we denote by $\sigma : V \otimes W \to W \otimes V$ the flip map.

(2) The following diagram commutes

$$\begin{array}{ccc} B \otimes B & \xrightarrow{m} & B \\ & {\scriptstyle \varepsilon\otimes\varepsilon}\searrow & \downarrow{\scriptstyle \varepsilon} \\ & & k \end{array}$$

(3) The following diagram commutes

$$\begin{array}{ccc} & k & \\ {\scriptstyle u}\swarrow & & \searrow{\scriptstyle u\otimes u} \\ B & \xrightarrow[\Delta]{} & B \otimes B \end{array}$$

(4) The composit is the identity

$$k \xrightarrow{u} B \xrightarrow{\varepsilon} k$$

2.3 Hopf Algebra

A Hopf algebra is a Bialgebra H together with a linear map $S : H \to H$ called the antipode, satisfying

$$\varepsilon(h)1 = S(h_{(1)})h_{(2)} = h_{(1)}S(h_{(2)})$$

for every $h \in H$. [8]

3 Quantum Groups

Let $\mathfrak{g}$ be a Lie algebra over a field k. Its formal deformation is a Lie algebra $\mathfrak{g}_\hbar$ over the ring $k[[\hbar]]$ together with an isomorphism $\mathfrak{g}_\hbar/\mathfrak{g}_\hbar\hbar \cong \mathfrak{g}$ of Lie algebra such that there is an isomorphism of $k[[\hbar]]$-modules $\mathfrak{g}_\hbar \cong \mathfrak{g}[[\hbar]]$. Choosing such an isomorphism we obtain $k[[\hbar]]$-linear Lie algebra structure on $\mathfrak{g}[[\hbar]]$ with a bracket

$$[-, -] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}[[\hbar]]$$

which coincides with the original bracket when $\hbar = 0$.
Some important results due to Nijenhius , Hoschild and Gertenhaber is that a finite dimentional simple Lie algebra $\mathfrak{g}$ has no nontrivial formal deformation [9] neither does it's Unversal envelpoing algebra $U(\mathfrak{g})$ admits a nontrivial formal deformation as an algebra [5]. But $U(\mathfrak{g})$ admits a unique nontrivial deformation as a bialgebra [4].

4 Quantization of the Universal Enveloping algebra $U(\mathfrak{sl}_2)$

Recall that the classical universal enveloping algebra has generators 1 and e, f, h with the following relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h$$

which admits a natural hopf algebra structure,

$$\Delta(h) = h \otimes 1 + 1 \otimes h,\ \Delta(e) = e \otimes 1 + 1 \otimes e,\ \Delta(f) = f \otimes 1 + 1 \otimes f$$
$$\varepsilon(1) = 1,\ \varepsilon(e) = 0\ \varepsilon(f) = 0$$
$$S(e) = -e \quad S(f) = -f,\ S(h) = -h$$

wherw Δ, ε, S is, respectively, the comultiplication, counit and antipode. The natural way to quantize this universal enveloping algebra is to consider the generators as $E, F, q^{\frac{H}{2}}, q^{-\frac{H}{2}}$ with the following relations. We will write $q^{\frac{H}{2}} = K$ from now on.

Definition 4.1. $U_q(\mathfrak{sl}_2)$ is defined as the noncommutative algebra generated by 1 and E, F, K, K^{-1} with the following relations.

$$KK^{-1} = K^{-1}K$$
$$KEK^{-1} = q^2 E$$
$$KFK^{-1} = q^{-2} F$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}$$

This forms a Hopf algebra structure with

$$\Delta(K) = K \otimes K,\ \Delta(K^{-1}) = K^{-1} \otimes K^{-1}$$
$$\Delta(E) = E \otimes 1 + K \otimes E,\ \Delta(F) = F \otimes K^{-1} + 1 \otimes F$$
$$\varepsilon(K) = \varepsilon(K^{-1}) = 1,\ \ \varepsilon(E) = \varepsilon(F) = 0$$
$$S(K) = K^{-1},\ S(K^{-1}) = K$$
$$S(E) = -qE,\ S(F) = -q^{-1}F$$

So far we have been working on the fraction field $\mathbb{Q}(q)$ of the polynomial ring $\mathbb{Q}[q]$, more specifically on the subring $\mathbb{Q}[q, q^{-1}]$ of $\mathbb{Q}(q)$. To find the classical limit we have to work over the field of complex number of the polynomial ring $\mathbb{C}[[\hbar]]$. Now, there exists a ring homomorphism $\phi : \mathbb{Q}[q, q^{-1}] \to \mathbb{C}[[\hbar]]$ by $\phi(q) = e^{\hbar/2}$ and $\phi(q^{-1}) = e^{-\hbar/2}$. Now we are ready to work with

$U_\hbar(\mathfrak{sl}_2)$. The relations here are,

$$e^{\hbar H/2} E e^{-\hbar H/2} = e^{\hbar} E$$
$$e^{\hbar H/2} F e^{-\hbar H/2} = e^{-\hbar} F$$
$$[E, F] = \frac{e^{\hbar H/2} - e^{-\hbar H/2}}{e^{\hbar/2} - e^{-\hbar/2}}$$

Proposition 4.1. *There is an isomorphism of Hopf algebras* $U_\hbar(\mathfrak{sl}_2)/\hbar U_\hbar(\mathfrak{sl}_2) \cong U(\mathfrak{sl}_2)$ *which sends* $(E, F, H) \mapsto (e, f, h)$

Proof. Since we work on the ring of formal power series in $\hbar$, $\mathbb{C}[[\hbar]]$ we can expand the exponential as,

$$e^{\hbar/2} = \sum_{n=0}^{\infty} \frac{(\hbar/2)^n}{n!} \quad e^{\frac{H\hbar}{2}} = \sum_{i=0}^{\infty} \frac{(\hbar H/2)^n}{n!}$$

Now we expand the relations of the algebra $U_q(\mathfrak{sl}_2)$ in $\hbar$. Using the Baker–Campbell–Hausdorff adjoint expansion

$$KEK^{-1} = e^{\hbar H/2} E e^{-\hbar H/2} = \left(1 + \frac{\hbar}{2} H + O(h^2)\right) E \left(1 - \frac{\hbar}{2} H + O(\hbar^2)\right)$$
$$= E + \frac{\hbar}{2}[HE - EH] + O(\hbar^2)$$

now expanding the right hand side

$$q^2 E = e^{\hbar} E = \left(1 + \hbar + O(h^2)\right) E = E + \hbar E + O(\hbar^2)$$

Equating both sides we get,

$$E + \frac{\hbar}{2}[HE - EH] + O(\hbar^2) = E + \hbar E + O(\hbar^2)$$
$$[H, E] = 2E + O(\hbar)$$

Now quotenting out by $\hbar U_q(\mathfrak{sl}_2)$ is the same as setting $\hbar = 0$. Then we find the classical relation,

$$[H, E] = 2E$$

Similarly

$$[H, F] = -2F$$

For $\frac{K - K^{-1}}{q - q^{-1}}$,

$$[E, F] = \frac{e^{\hbar H/2} - e^{-\hbar H/2}}{e^{\hbar/2} - e^{-\hbar/2}} = \frac{\sinh(\hbar H/2)}{\sinh(\hbar/2)} = \frac{\hbar H/2 + O(\hbar^3)}{\hbar/2 + O(\hbar^3)} = \frac{H/2 + O(\hbar^2)}{1/2 + O(\hbar^2)} = H.$$
$$\implies [E, F] = H.$$

For the hopf algebra isomorphisms, notice that

$$K = e^{\hbar H/2} = 1 + \frac{\hbar}{2} H + O(\hbar^2), \quad \Delta(K) = e^{\hbar \Delta(H)/2} = K \otimes K$$

Now,

$$\begin{aligned}
\log(e^{\hbar\Delta(H)/2}) &= \frac{\hbar}{2}\Delta(H) = \log(K\otimes K)\\
&\frac{\hbar}{2}\Delta(H) = \log(K\otimes 1) + \log(1\otimes K)\\
&\therefore \Delta(H) = \left(\frac{1}{\hbar/2}\log(K)\right)\otimes 1 + 1\otimes\left(\frac{1}{\hbar/2}\log(K)\right) = H\otimes 1 + 1\otimes H
\end{aligned}$$

Now for the other generators:

$$\Delta(E) = E\otimes K + 1\otimes E = E\otimes e^{\hbar H/2} \xrightarrow{\hbar\to 0} E\otimes 1 + 1\otimes E.$$
$$\Delta(F) = F\otimes 1 + K^{-1}\otimes F = F\otimes 1 + e^{-\hbar H/2}\otimes F \xrightarrow{\hbar\to 0} F\otimes 1 + 1\otimes F.$$

These also map directly to the coproducts for the classical generators e and f.

For the counit, we have $\varepsilon(E) = 0$, $\epsilon(F) = 0$, and $\epsilon(K) = 1$. The first two relations map directly to their classical counterparts $\varepsilon(e) = 0$ and $\varepsilon(f) = 0$. The relation $\varepsilon(K) = 1$ means $\varepsilon(e^{\hbar H/2}) = e^{\hbar\varepsilon(H)/2} = 1$. For this formal power series to be 1, the exponent must be zero, which implies $\varepsilon(H) = 0$. This corresponds to the classical counit $\varepsilon(h) = 0$.

Finally, for the antipode, in the limit $\hbar \to 0$:

$$\begin{aligned}
S(H) &= -H\\
S(E) &= -EK^{-1} = -Ee^{-\hbar H/2} \xrightarrow{\hbar\to 0} -E.\\
S(F) &= -KF = -e^{\hbar H/2}F \xrightarrow{\hbar\to 0} -F.
\end{aligned}$$

These map precisely to the classical antipodes $S(h) = -h$, $S(e) = -e$, and $S(f) = -f$.

Since the algebra relations and all the Hopf algebra structures of $U_q(\mathfrak{sl}_2)$ reduce to their classical counterparts the isomorphism holds as an isomorphism of Hopf algebras. □

Lemma 4.2. *The algebra $U_q(\mathfrak{sl}_2)$ is spanned by the monomials $F^sK^nE^r$ with $s, n, r \in \mathbb{Z}$ and $r, s > 0$.*

Theorem 4.3. *The monomials $F^sK^nE^r$ with $s, n, r \in \mathbb{Z}$ and $r, s > 0$ form a basis of $U_q(\mathfrak{sl}_2)$*

5 Representations of $U_q(\mathfrak{sl}_2)$

Our aim in this section is to find all the finite-dimensional representations of $U_q(\mathfrak{sl}_2)$ and determine the center of $U_q(\mathfrak{sl}_2)$. In both cases the result varies depending on whether q is a root of unity or not. From now on we will only consider the case where q is not a root of unity unless stated otherwise. In this case the result is quite similar to the case for $U(\mathfrak{sl}_2)$. [6]

Proposition 5.1. *Let M be a finite dimensional $U(\mathfrak{sl}_2)$-module. There are integers $r, s > 0$ with $E^rM = 0$ and $F^sM = 0$.*

If M is a $U(\mathfrak{sl}_2)$-module then we define

$$M_\lambda = \{m \in M \mid Km = \lambda m\}$$

We call M_λ the weight space of M and λ's the weights of M.

Lemma 5.2. *we have*

$$EM_\lambda \subset M_{q^2\lambda} \quad \text{and } FM_\lambda \subset M_{q^{-2}\lambda}$$

Proposition 5.3. *Let M be a finite dimensional $U(\mathfrak{sl}_2)$-module. Then M is a direct sum of its weight spaces. All weights of M have the form $\pm q^a$ with $a \in \mathbb{Z}$.*

For each $\lambda \in k$ there is an inifnite dimensional $U_q(\mathfrak{sl}_2)$-module $M(\lambda)$ with basis $m_0, m_1, \ldots$ such that for all i

$$Km_i = \lambda q^{-2i} m_i$$

$$Fm_i = m_{i+1}$$

$$Em_i = \begin{cases} 0 & \text{if } i = 0 \\ [i]\frac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q - q^{-1}} m_{i-1} & otherwise \end{cases}$$

The verma modules is constructed as

$$M(\lambda) = U_q(\mathfrak{sl}_2)/(U_q(\mathfrak{sl}_2)E + U_q(\mathfrak{sl}_2)(K - \lambda))$$

Proposition 5.4. *Let $\lambda \in k, \lambda \neq 0$. If $\lambda \neq \pm q^n$ for all integers $n \geq 0$, then the $U_q(\mathfrak{sl}_2)$-module is simple. If $\lambda = \pm q^n$ for some integer $n \geq 0$, then the m_i with $i \geq n+1$ span a submodule of $M(\lambda)$ isomprphic to $M(q^{-2(n+1)}\lambda)$; this is the only submodule of $M(\lambda)$ different from* 0 *and $M(\lambda)$.*

Theorem 5.5. *There are for each integer $N \geq 0$ a simple U-module $L(N, +)$ with basis $m_0, m_1, \ldots, m_n$ and a simple $U_q(\mathfrak{sl}_2)$-module $L(n, -)$ with basis $m'_0, m'_1, \ldots, m'_n$ such that for all i $(0 \leq i \leq n)$:*

$$Km_i = q^{n-2i} m_i, \qquad Km'_i = -q^{n-2i} m'_i,$$

$$Fm_i = \begin{cases} m_{i+1}, & \text{if } i < n, \\ 0, & \text{if } i = n, \end{cases} \qquad Fm'_i = \begin{cases} m'_{i+1}, & \text{if } i < n, \\ 0, & \text{if } i = n, \end{cases}$$

$$Em_i = \begin{cases} [i][n+1-i]\, m_{i-1}, & \text{if } i > 0, \\ 0, & \text{if } i = 0, \end{cases} \qquad Em'_i = \begin{cases} -[i][n+1-i]\, m'_{i-1}, & \text{if } i > 0, \\ 0, & \text{if } i = 0. \end{cases}$$

Each simple U-module of dimension $n+1$ is isomorphic to $L(n, +)$ or to $L(n, -)$.

There is a casimir element similiar to the classical case defined as the following,

Proposition 5.6. *The element*

$$C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q - q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q - q^{-1})^2}$$

belongs to the center of $U_q(\mathfrak{sl}_2)$

Lemma 5.7. *Let L and L' be finite dimensional $U_q(\mathfrak{sl}_2)$-module. If C_q acts on L by the same factor as on L', then L is isomorphic to L'.*

6 The universal R-matrix of $U_q(\mathfrak{sl}_2)$

6.1 Historical Context

The modern study of the R-matrix and the Quantum Yang–Baxter Equation emerged from efforts to solve exactly integrable models in statistical mechanics and quantum field theory. The equation first appeared implicitly in C. N. Yang's analysis of the 1D many-body delta-function Bose gas.[11] Then R. J. Baxter discovered a similar condition while solving the eight-vertex model in lattice statistical mechanics, introducing what became known as the star–triangle relation.[1]

The explicit formulation of the quantum Yang–Baxter equation as an operator identity for an $R-$ matrix was introduced in the 1980s, when Faddeev, Sklyanin, and Takhtajan connected it to the Quantum Inverse Scattering Method and integrable systems[3]. Independently, Drinfeld and Jimbo recognized that solutions of the QYBE encode a rich Hopf algebraic structure, leading to the introduction of quantum groups as deformations of universal enveloping algebras[2]. Since then, the R-matrix has become a central object to mathematical physics, representation theory, and low-dimensional topology.

6.2 Construction

In the classical universal enveloping algebra $U(\mathfrak{sl}_2)$, the comultiplication is given by

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

$\forall x \in U(\mathfrak{sl}_2)$. Now if we define a new map, $P : M \otimes N \to N \otimes M$ that swaps the two components of the tensor product $P(a \otimes b) = b \otimes a$, the result remains the same

$$P \circ \Delta = \Delta$$

This flip map P is a $U(\mathfrak{sl}_2)$-module isomorphism. In the quantum case it's not the case, in fact quantum groups were invented so as to have this defect. There is a way of correcting the non-cocommutativity of $U_q(\mathfrak{sl}_2)$ and the correction terms have properties that people were looking for.
To fix this issue we have to find an element $R \in U_q(\mathfrak{sl}_2)^{2\otimes}$ such that

$$R \circ P \circ \Delta(u) = \Delta(u) \circ R \quad for\ all\ u \in U_q(\mathfrak{sl}_2)$$

We will do it in a few steps.
We have three coproducts on $U_q(\mathfrak{sl}_2)$:

$$\begin{aligned} \Delta(K) &= K \otimes K, & \Delta(E) &= E \otimes 1 + K \otimes E, & \Delta(F) &= F \otimes K^{-1} + 1 \otimes F, \\ \Delta^{op}(K) &= K \otimes K, & \Delta^{op}(E) &= E \otimes K + 1 \otimes E, & \Delta^{op}(F) &= F \otimes 1 + K^{-1} \otimes F, \\ \Delta'(K) &= K \otimes K, & \Delta'(E) &= E \otimes 1 + K^{-1} \otimes E, & \Delta'(F) &= F \otimes K + 1 \otimes F. \end{aligned}$$

The coproduct Δ' is obtained from Δ by twisting with the anti–involution

$$\tau(K) = K^{-1}, \qquad \tau(E) = E, \qquad \tau(F) = F,$$

so that

$$\Delta'(u) = (\tau \otimes \tau)\big(\Delta(\tau(u))\big).$$

Instead of trying to directly solve

$$R\Delta(u)R^{-1} = \Delta^{op}(u),$$

we define two objects $\Theta \in U^{\otimes 2}$ and $\Psi \in U^{\otimes 2}$ such that

$$\Theta^{-1}\,\Delta(u)\,\Theta = \Delta'(u) \quad \Psi^{-1}\,\Delta'(u)\,\Psi = \Delta^{op}(u)$$

and we set

$$R = \Theta\Psi$$

gives

$$R^{-1}\,\Delta(u)\,R = \Psi^{-1}\Theta^{-1}\,\Delta(u)\,\Theta\Psi = \Psi^{-1}\,\Delta'(u)\,\Psi = \Delta^{op}(u).$$

We will construct both Θ and Φ.

<u>Construction of Θ:</u> Θ will be a infinite sum of the form

$$\Theta = \sum_{n=0}^{\infty} a_n F^n \otimes E^n$$

. We have to find the coefficients a_n's which we will do by inserting this expression in the relation $\Theta^{-1}\,\Delta(u)\,\Theta = \Delta'(u)$. So,

$$\begin{aligned}
&\Theta\Delta'(E) = \Delta(E)\Theta \\
\implies &\Big(\sum a_n F^n \otimes E^n\Big)\big(E \otimes 1 + K^{-1} \otimes E\big) = (E \otimes 1 + K \otimes E)\Big(\sum a_n F^n \otimes E^n\Big) \\
\implies &\sum a_n (F^n E \otimes E^n + F^n K^{-1} \otimes E^{n+1}) = \sum a_n (EF^n \otimes E^n + KF^n \otimes E^{n+1}) \\
\implies &\sum a_n [E, F^n] \otimes E^n = \sum a_n (F^n K^{-1} - KF^n) \otimes E^{n+1} \\
\therefore \quad & a_{n+1}[E, F^{n+1}] = a_n(F^n K^{-1} - KF^n)
\end{aligned}$$

Now we know that

$$[E, F^{n+1}] = [n+1]_q F^n \frac{Kq^{-n} - K^{-1}q^n}{q - q^{-1}}$$

and

$$F^n K^{-1} - KF^n = F^n(K^{-1} - q^{-2n}K) \tag{1}$$

Putting them altogether we get,

$$\begin{aligned}
a_{n+1}[n+1]_q F^n \frac{Kq^{-n} - K^{-1}q^n}{q - q^{-1}} &= a_n F^n (K^{-1} - q^{-2n}K) \\
a_{n+1}[n+1]_q F^n \frac{K^{-1} - q^{-2n}K}{q^{-1} - q} q^n &= a_n F^n (K^{-1} - q^{-2n}K) \\
a_{n+1} &= \frac{(q^{-1} - q)q^{-n}}{[n+1]_q} a_n
\end{aligned}$$

We take $a_0 = 1$, then

$$\Theta = \sum_{n=0}^{\infty} \frac{(q^{-1}-q)^n q^{-n(n-1)/2}}{[n]_q!} F^n \otimes E^n$$

The same Θ satisfies $\Theta\Delta'(F) = \Delta(F)\Theta$ and $\Theta\Delta'(K) = \Delta(K)\Theta$.

Construction of Ψ: Ψ will depend only on K. We will take two $U_q(\mathfrak{sl}_2)$-modules V_1, V_2 and $u_m \in V_1$ and $v_n \in V_2$ be two K-eigenvectors with eigenvalue q^m and q^n. So,

$$Kv_n = q^n v_n \quad Ku_m = q^m u_m$$

Thus $v_n \otimes u_m$ is a eigenvector for $\Delta(K) = K \otimes K$ with eigenvalue q^{n+m}. Let

$$\Psi(v_n \otimes u_m) = \psi(n,m) v_n \otimes u_m$$

Then we require

$$\begin{aligned} &\Psi\Delta^{\mathrm{op}}(E)(v_n \otimes u_m) = \Delta'(E)\Psi(v_n \otimes u_m) \\ \Longrightarrow\ &\Psi(q^m E v_n \otimes u_m + v_n \otimes E u_m) = \psi(n,m)(E \otimes 1 + K^{-1} \otimes E)(v_n \otimes u_m) \\ \Longrightarrow\ &q^m \psi(n+2,m) E v_n \otimes u_m + \psi(n,m+2) v_n \otimes E u_m = \psi(n,m) E v_n \otimes u_m + q^{-n}\psi(n,m) v_n \otimes E u_m \end{aligned}$$

$$\therefore \quad \psi(n+2,m) = q^{-m}\psi(n,m) \quad \text{and} \quad \psi(n,m+2) = q^{-n}\psi(n,m)$$

Example 1. *Let $V_2 = L(1,+)$ be a 2-dimensional module and $V_3 = L(2,+)$ a 3-dimensional module. Then for V_2 with basis $\{m_0, m_1\}$*

$$\begin{gathered} Km_0 = qm_0 \quad Km_1 = q^{-1}m_1 \\ Fm_0 = m_1 \quad Fm_1 = 0 \\ Em_0 = 0 \quad Em_1 = m_0 \end{gathered}$$

and for V_3 with basis $\{v_0, v_1, v_2\}$

$$\begin{gathered} Kv_0 = q^2 v_0 \quad Kv_1 = v_1 \quad Kv_2 = q^{-2} v_2 \\ Fv_0 = v_1 \quad Fv_1 = v_2 \quad Fv_2 = 0 \\ Ev_0 = 0 \quad Ev_1 = [2]v_0 \quad Ev_2 = [2]v_1 \end{gathered}$$

Now we have

$$\Theta = 1 + (q - q^{-1}) F \otimes E$$

The basis for $V_2 \otimes V_3$ is

$$\{m_0 \otimes v_0,\ m_0 \otimes v_1,\ m_0 \otimes v_2,\ m_1 \otimes v_0,\ m_1 \otimes v_1,\ m_1 \otimes v_2\}$$

Now,

$$\Theta(m_0 \otimes v_0) = m_0 \otimes v_0 + (q - q^{-1})Fm_0 \otimes Ev_0 = m_0 \otimes v_0$$
$$\Theta(m_0 \otimes v_1) = m_0 \otimes v_1 + (q - q^{-1})Fm_0 \otimes Ev_1 = m_0 \otimes v_1 + (q - q^{-1})[2]m_1 \otimes v_0$$
$$\Theta(m_0 \otimes v_2) = m_0 \otimes v_2 + (q - q^{-1})Fm_0 \otimes Ev_2 = m_0 \otimes v_2 + (q - q^{-1})[2]m_1 \otimes v_1$$
$$\Theta(m_1 \otimes v_0) = m_1 \otimes v_0 + (q - q^{-1})Fm_1 \otimes Ev_0 = m_1 \otimes v_0$$
$$\Theta(m_1 \otimes v_1) = m_1 \otimes v_1 + (q - q^{-1})Fm_1 \otimes Ev_1 = m_1 \otimes v_1$$
$$\Theta(m_1 \otimes v_2) = m_1 \otimes v_2 + (q - q^{-1})Fm_1 \otimes Ev_2 = m_1 \otimes v_2$$

Then

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & q^2 - q^{-2} & 0 & 1 & 0 & 0 \\ 0 & 0 & q^2 - q^{-2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Now Ψ acts on the tensor product of weight vectors $w_\lambda \otimes w_\mu$ as:

$$\Psi(w_\lambda \otimes w_\mu) = q^{\lambda\mu/2}(w_\lambda \otimes w_\mu)$$

Now we will calculate this scalar factor for each basis vector of $V_2 \otimes V_3$ using the weights for $\lambda \in \{1, -1\}$ and $\mu \in \{2, 0, -2\}$.

1. $m_0 \otimes v_0$: $(\lambda, \mu) = (1, 2) \implies q^{1\cdot 2/2} = q^1$
2. $m_0 \otimes v_1$: $(\lambda, \mu) = (1, 0) \implies q^{1\cdot 0/2} = q^0 = 1$
3. $m_0 \otimes v_2$: $(\lambda, \mu) = (1, -2) \implies q^{1\cdot(-2)/2} = q^{-1}$
4. $m_1 \otimes v_0$: $(\lambda, \mu) = (-1, 2) \implies q^{(-1)\cdot 2/2} = q^{-1}$
5. $m_1 \otimes v_1$: $(\lambda, \mu) = (-1, 0) \implies q^{(-1)\cdot 0/2} = q^0 = 1$
6. $m_1 \otimes v_2$: $(\lambda, \mu) = (-1, -2) \implies q^{(-1)\cdot(-2)/2} = q^1$

This gives us a purely diagonal matrix:

$$\Psi_{V_2,V_3} = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}$$

So,

$$R_{V_2,V_3} = \Theta_{V_2,V_3}\Psi_{V_2,V_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & q^2-q^{-2} & 0 & 1 & 0 & 0 \\ 0 & 0 & q^2-q^{-2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}$$

$$= \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 & 0 & 0 \\ 0 & q^2-q^{-2} & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & q^{-1}(q^2-q^{-2}) & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}$$

7 Hisenberg XXZ Model

Historically the motivation behind developing Quantum Groups was because of their appearance in Quantum Integrable Systems , for example Spin chains. In this section we will talk about the Hisenberg XXZ model where the symmetry is governed by the quantum group $U_q(\mathfrak{sl}_2)$.

Let's consider a chain of N spins-$\frac{1}{2}$ located at lattice points $i = 1, 2, \ldots, N$ with nearest-neighbour interactions. The Hamiltonian of the XXZ model is

$$H_{XXZ} = J \sum_{i=1}^{N-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \delta\, \sigma_i^z \sigma_{i+1}^z \right),$$

where $\sigma_i^{x,y,z}$ are the Pauli matrices acting on i, J is the coupling constant, and δ is a real parameter controlling the anisotropy between the interactions in the xy-plane and the z-direction. The isotropic Heisenberg model corresponds to $\delta = 1$ and the anisotropic Heisenberg model corresponds to $\delta \neq 1$.

We introduce a deformation parameter q defined by

$$\delta = \frac{1}{2}\left(q + q^{-1}\right),$$

so that $\delta = 1$ corresponds to $q = 1$. This parameter q will later be seen to coincide with the q appearing in the quantum group $U_q(\mathfrak{sl}_2)$.

Let E, F, K, K^{-1} be the generators of $U_q(\mathfrak{sl}_2)$ satisfying the relations we studied before. Now each spin site carries a 2-dimensional spin-$\frac{1}{2}$ representation $V_{1/2}$ of $U_q(\mathfrak{sl}_2)$ which is the fundamental represenation of $U_q(\mathfrak{sl}_2)$. By iterating the coproduct Δ we can obtain the entire spin chain so that $U_q(\mathfrak{sl}_2)$ acts globally on $V^{\otimes N}$. The Hamiltonian then commutes with the global action,

$$[H_{XXZ}, \Delta^{(N)}(E)] = [H_{XXZ}, \Delta^{(N)}(F)] = [H_{XXZ}, \Delta^{(N)}(K)] = 0,$$

implying that $U_q(\mathfrak{sl}_2)$ plays the role of a deformed symmetry group, which in the classical case would be $U(\mathfrak{sl}_2)$ as $q \to 1$.

The integrability of the XXZ model is based on the existence of an R-matrix satisfying the quantum Yang–Baxter equation,

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v),$$

where u and v are spectral parameters and the subscripts denote the tensor components on which the operator acts. The R-matrix encodes the two-body scattering data of the system.

For the fundamental representation of $U_q(\mathfrak{sl}_2)$, the R-matrix takes the following form [10]

$$R(u) = \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh(u) & \sinh(\eta) & 0 \\ 0 & \sinh(\eta) & \sinh(u) & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix},$$

where $q = e^{\eta}$. This R-matrix can be obtained from the universal R-matrix of $U_q(\mathfrak{sl}_2)$ constructed earlier by evaluation in the spin-$\frac{1}{2}$ representation. It satisfies both the QYBE and the intertwining relation

$$R\,\Delta(x) = \Delta^{\mathrm{op}}(x)\,R, \quad \forall\, x \in U_q(\mathfrak{sl}_2),$$

At $q = 1$, the model reduces to the isotropic Heisenberg chain, whose symmetry algebra is the classical $U(\mathfrak{sl}_2)$ Lie algebra. For $q \neq 1$, the symmetry is deformed to $U_q(\mathfrak{sl}_2)$, but integrability is preserved, and the exact spectrum can be obtained by the algebraic Bethe ansatz method.

Hence, the XXZ spin chain model provides a concrete physical model whose symmetries and integrability are governed by the abstract quantum group $U_q(\mathfrak{sl}_2)$.

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Signature: ______________________ 04-11-25

(Supervisor's Signature and Date)